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Entanglement in Holographic Theories

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Grant-in-Aid for Transformative Research Areas A
Extreme Universe

Contents

References on Holographic Entanglement Entropy

Reviews: ∗ Rangamani−TT, arXiv: 1609.01287 [Lect.Notes Phys. 931 (2017)]

∗ Van Raamsdonk, arXiv: 1609.00026 [TASI 2015, 297]

∗ Nishioka−Ryu−TT, arXiv:0905.0932, [J.Phys.A42:504008,2009]

Essay: ∗ TT, arXiv: 2506.06595 [Phys.Rev.Lett. 134 (2025) 24, 240001]

② AdS/BCFT [arXiv 1105.5165, 1108.5152]

(2-1) Boundary conformal field theory (BCFT)

(2-2) AdS/BCFT

(2-3) HEE in AdS/BCFT

(2-4) Holographic g-theorem

③ Moving Mirror and EE [arXiv 2011.12005, 2106.11179]

(3-1) BCFT description

(3-2) Calculating Entanglement Entropy

(3-3) Holographic Moving Mirror

① Holographic Entanglement Entropy

(1-1) Holographic Setup: AdS/CFT

[Maldacena 1997]

AdS/CFT

Massless (gapless) theory

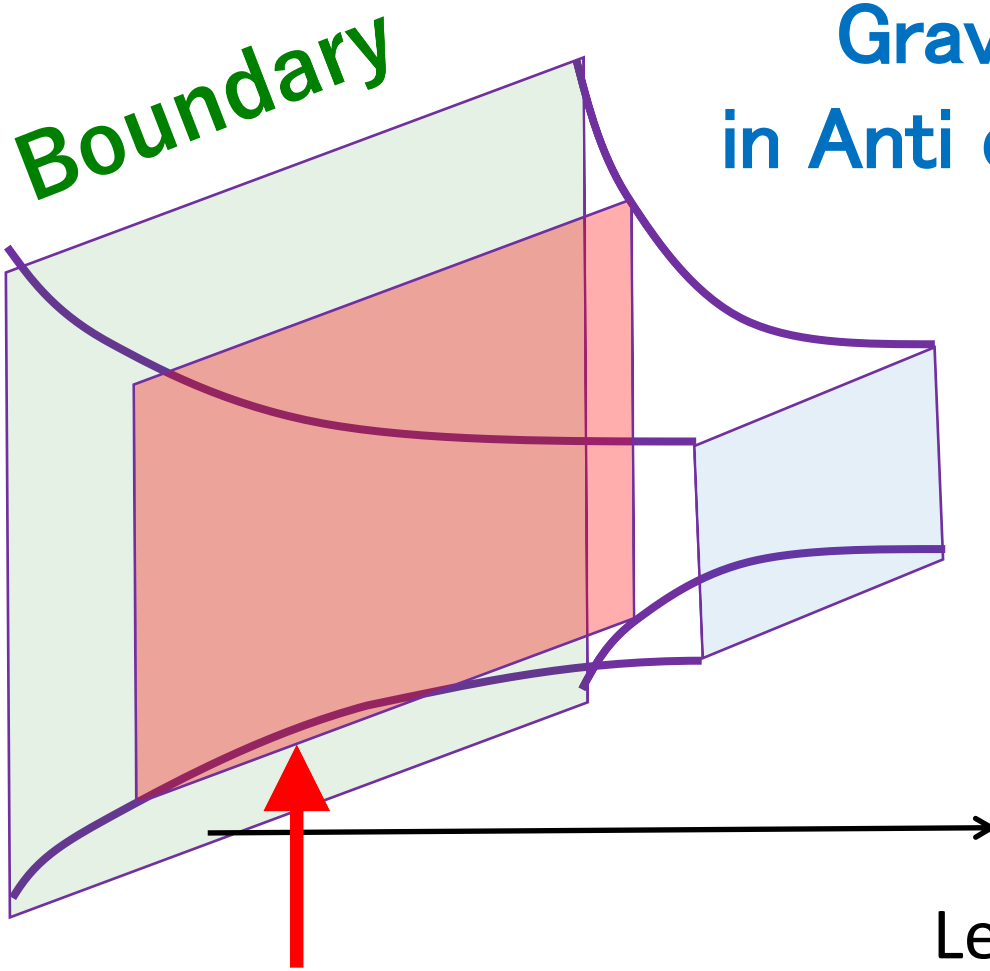
Gravity (String theory) on D+1 dim. AdS (anti de−Sitter space)	=	Conformal Field Theory (CFT) on D dim. Minkowski spacetime
⬇ **Classical limit**		⬇ **Large N + Strong coupling**
General relativity with $\Lambda < 0$		Strongly interacting Quantum Field Theories

Basic Principle

(Bulk−Boundary relation) : Partition Function

$$Z_{Gravity} = Z_{CFT}$$

Gravity
in Anti de-Sitter space (AdS)



AdS metric

$$ds^2 = R^2 \cdot \frac{dz^2 - dt^2 + \sum_{i=1}^{d} dx_i^2}{z^2}$$

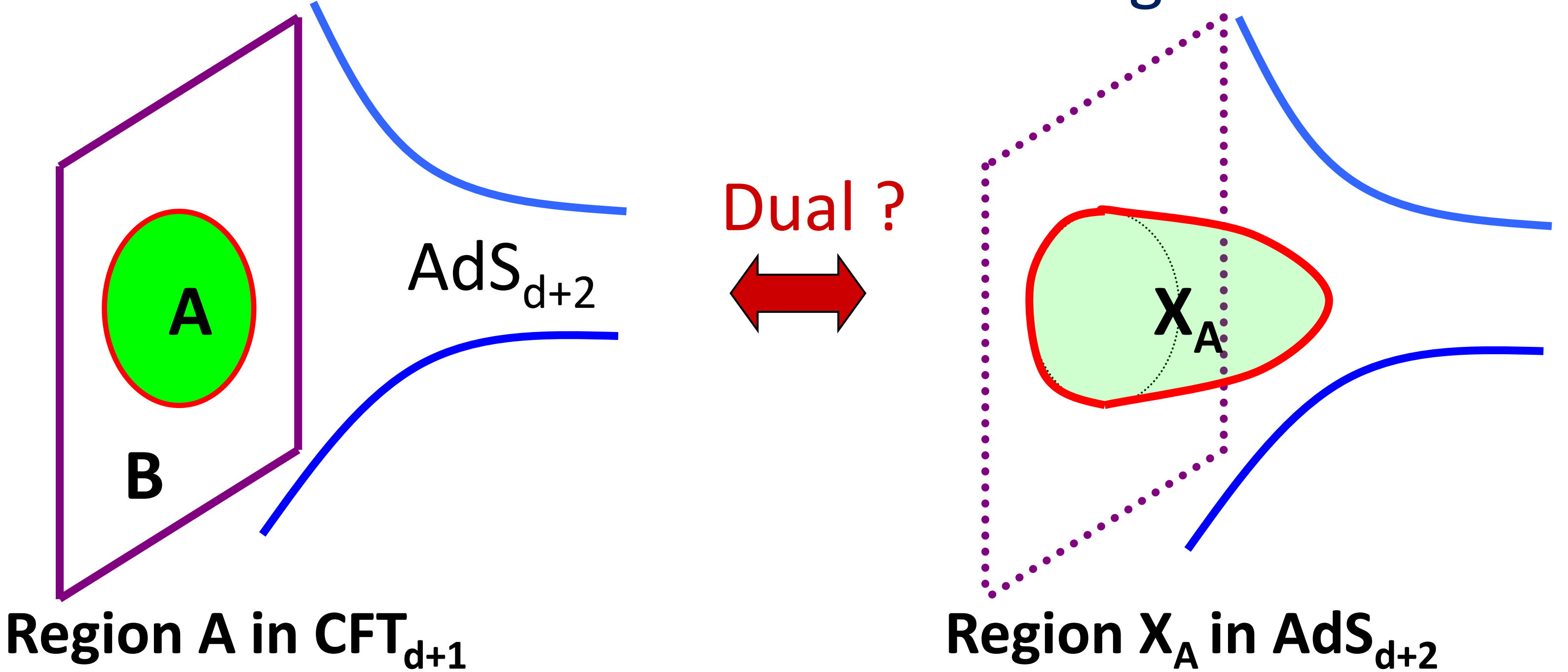
z

Length scale in CFT

z>ε

(UV cut off in CFT)

A Basic Question: Which region in the AdS does encode the `information in a certain region' of the CFT ?



➡ Consider the entanglement entropy S_A which measures the amount of information !

(1-2) EE in CFT and Replica Method

Replica Method

A basic method of calculating EE in QFTs is the **replica method**.

$$S_A = -\frac{\partial}{\partial n} \mathrm{Tr}_A (\rho_A)^n |_{n=1} = -\frac{\partial}{\partial n} \log \mathrm{Tr}_A (\rho_A)^n |_{n=1} \quad .$$

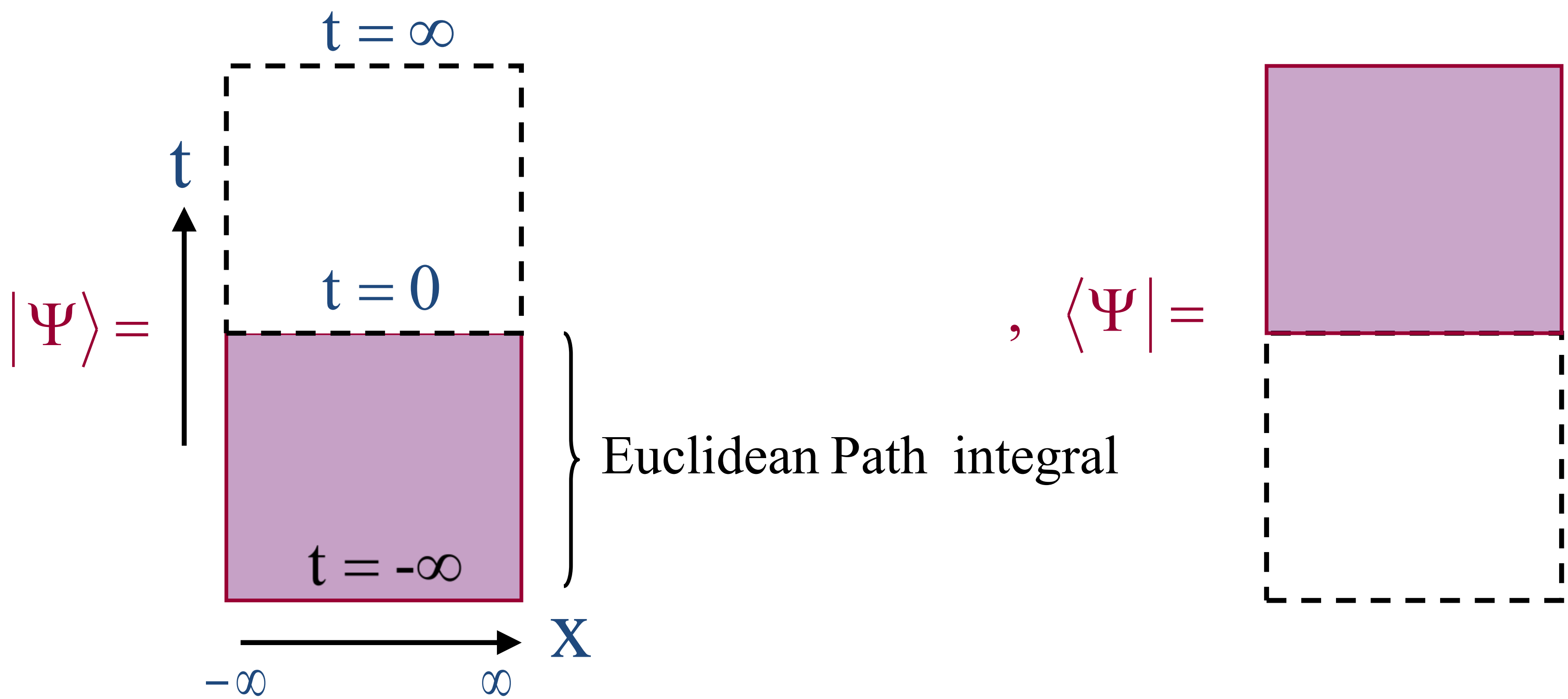
↑ **Compute this trace as a partition function**

Below we explain the replica method for two dimensional (2d) QFTs. Our main target is the analysis for 2d CFTs.

[Holzhey–Larsen–Wilczek 94,⋯, Calabrese–Cardy 04]

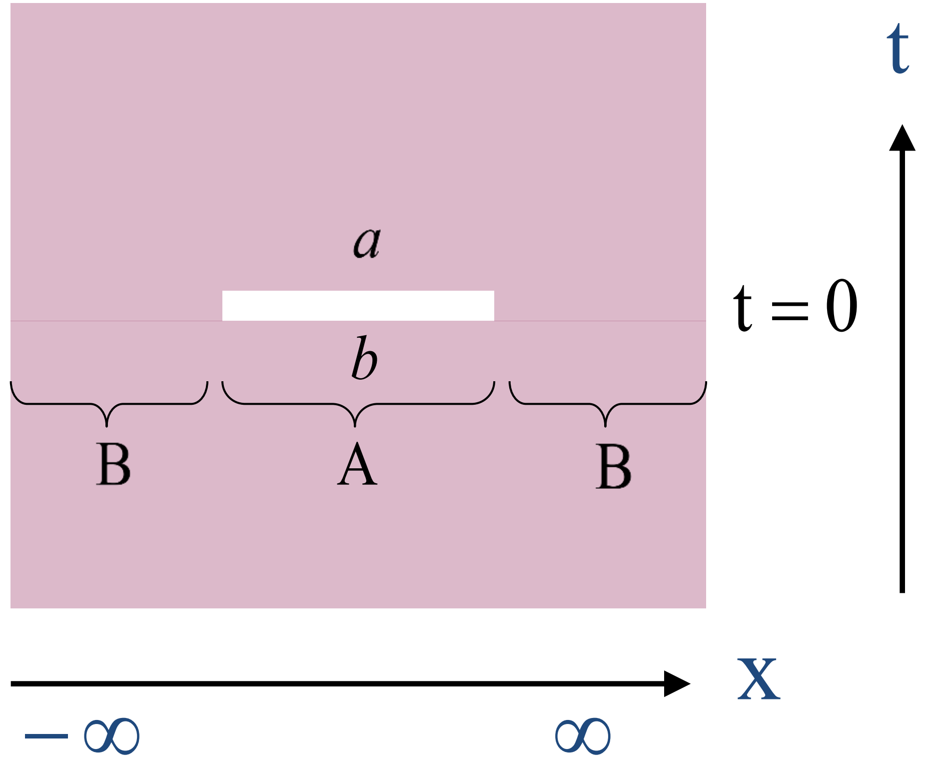
The replica method is also an important method to (often numerically) evaluate EE in higher dimensional QFTs.

In the path-integral formalism, the ground state wave function $|\Psi\rangle$ can be expressed in the path-integral formalism as follows:



We express the reduced density matrix $\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$:

$$[\rho_A]_{ab} =$$

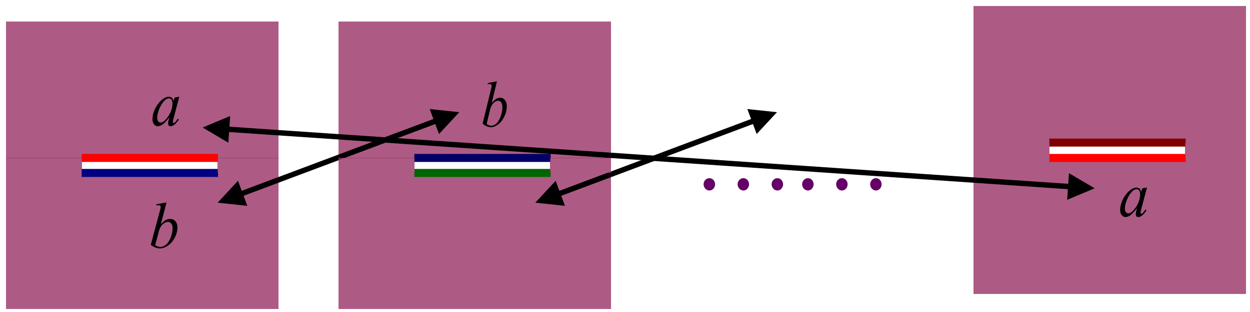


Finally, we obtain a path integral expression of the trace

$\mathrm{Tr}\left(\rho_A\right)^n = [\rho_A]_{ab}[\rho_A]_{bc}\cdots[\rho_A]_{ka}$ as follows:

Glue each boundaries successively.

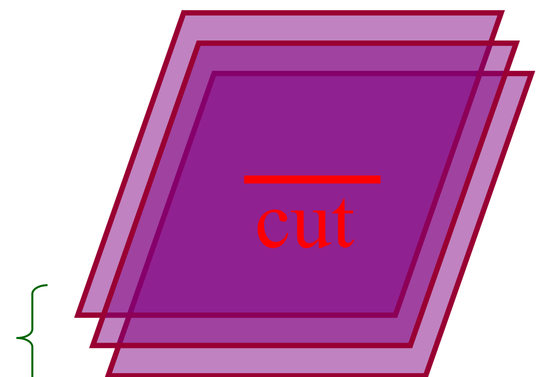
$\mathrm{Tr}\left(\rho_A\right)^n =$



= a path integral over

n-sheeted Riemann surface Σ_{n}

n sheets



In this way, we obtain the following representation

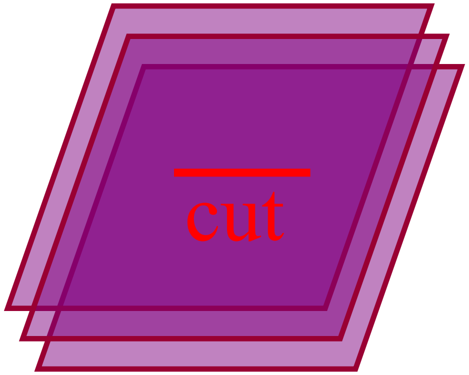
$$\mathrm{Tr}\left(\rho_A\right)^n = \frac{Z_n}{(Z_1)^n} \ ,$$

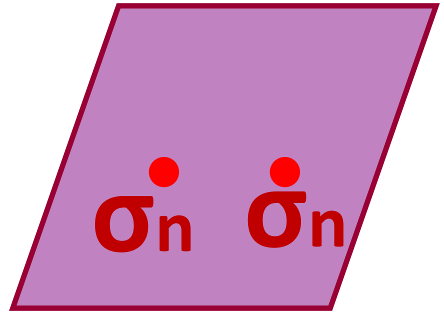
where Z_n is the partition function on the n-sheeted Riemann surface Σ_n.

Equally we can regard the partition function Z_n as that of a n-replicated CFT on a plane

Replicated fields

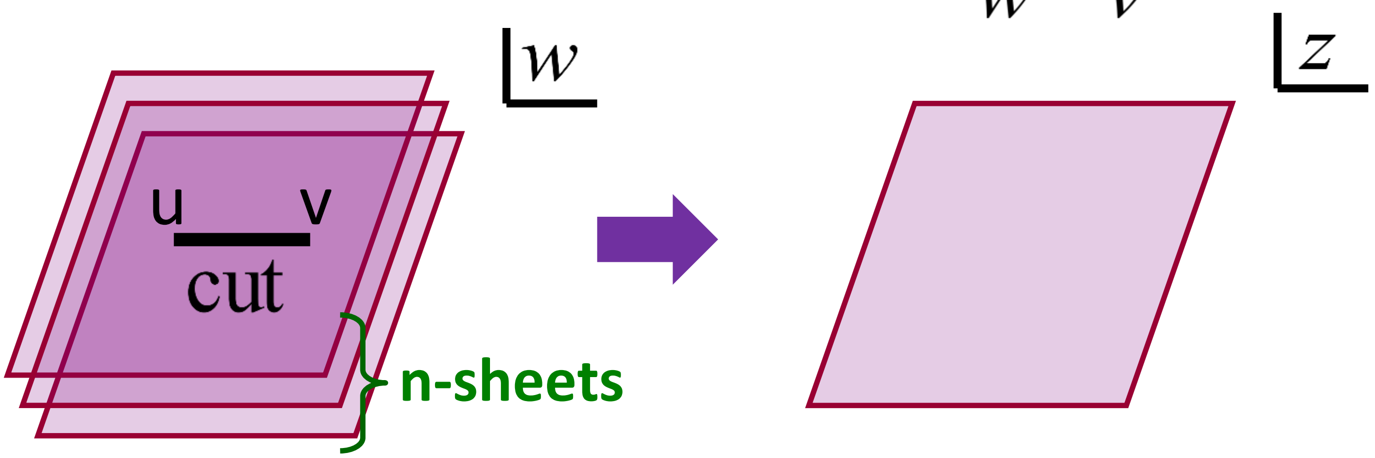
$$\phi \ \text{ on } \Sigma_{\mathrm{n}} \quad \longrightarrow \quad \phi_1, \phi_2, \cdots \phi_n \ \text{ on } \mathrm{C}$$





EE in 2d CFTs [Calabrese-Cardy 04]

Consider the conformal map: $z^n = \dfrac{w-u}{w-v}$.



$$T(w) = \left(\frac{dz}{dw}\right)^2 \underbrace{T(z)}_{=0} + \frac{c}{12}\underbrace{\{z,w\}}_{\text{Schwarzian derivative}} = \frac{c(1-n^{-2})}{24}\cdot\frac{(v-u)^2}{(w-u)^2(w-v)^2}.$$

Schwarzian derivative $(z'''z' - \frac{3}{2}z''^2)/z'^2$

$$\Rightarrow h_{\text{each sheet}} = \frac{c(1-n^{-2})}{24}, \qquad h_{\text{tot}} = nh_{\text{each sheet}} = \frac{c(n-1/n)}{24}.$$

Thus for general 2d CFTs with the central charge c, we obtain

$$\mathrm{Tr}\,(\rho_A)^n \propto (u-v)^{-4h_{tot}} = (u-v)^{-\frac{c}{6}(n-1/n)}.$$

In the end, we obtain

Renyi Entropy: $$S_A^{(n)} = \frac{c}{6}\left(1+\frac{1}{n}\right)\log\frac{l}{\varepsilon} \qquad (l \equiv v-u).$$

Entanglement Entropy: $$S_A = \frac{c}{3}\log\frac{l}{\varepsilon}\ .$$

[Holzhey-Larsen-Wilczek 94]

Note: the UV cut off ε is introduced.

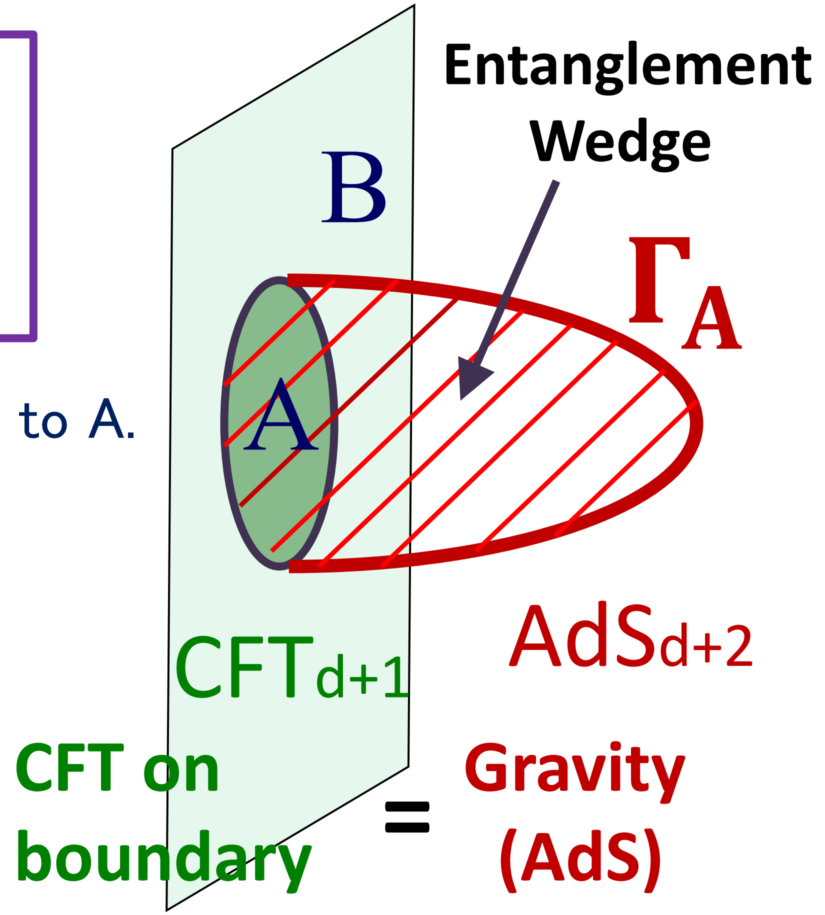
(1-3) HEE for Static Backgrounds

[Ryu−TT 2006]

For a static asymptotically AdS background, S_A can be computed from the minimal area surface Γ_A:

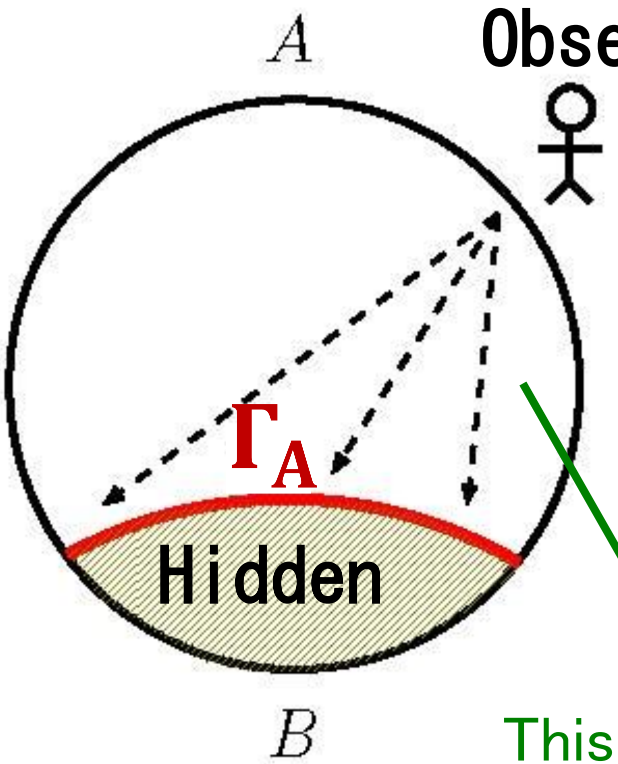
$$S_A = \min_{\Gamma_A}\left[\frac{\mathrm{Area}(\Gamma_A)}{4G_N}\right]$$

Note: $\partial\Gamma_A = \partial A$ and Γ_A is homologous to A.



This formula was later proved by Lewkowycz−Maldacena 2013 based on the bulk−bdy relation of AdS/CFT.

Intuitive Understanding of This Formula



Observer who cannot access B will find a "black hole" at Γ_A .

⇒ This BH entropy is the HEE !

This white region is accessible for an observer in A.

⇒ This is called **entanglement wedge**.

Leading divergence and Area law

For a generic choice of γ_{A}, a basic property of AdS gives

$$\mathrm{Area}(\gamma_{\mathrm{A}}) \sim R^{d} \cdot \frac{\mathrm{Area}(\partial\gamma_{\mathrm{A}})}{\varepsilon^{d-1}} + (\text{subleading terms}),$$

where R is the AdS radius.

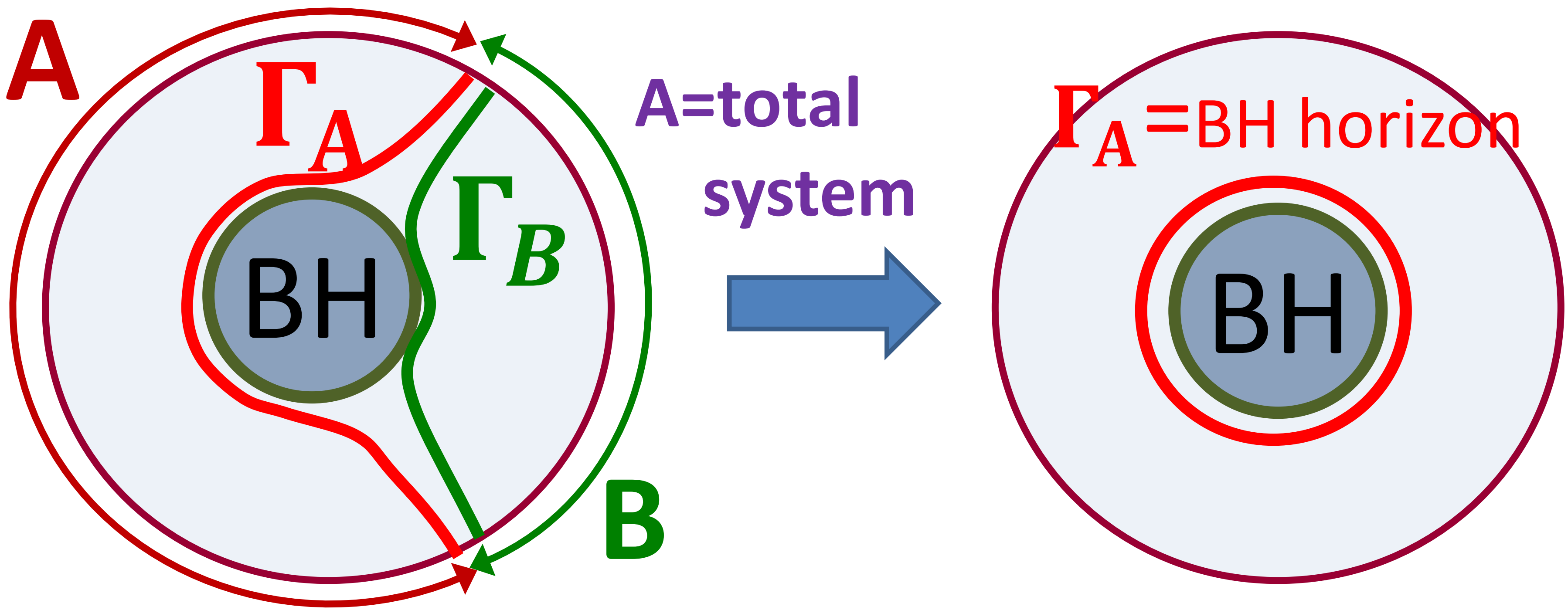
Because $\partial\gamma_{\mathrm{A}} = \partial \mathrm{A}$, we find

$$S_{A} \sim \frac{\mathrm{Area}(\partial \mathrm{A})}{\varepsilon^{d-1}} + (\text{subleading terms}).$$

This agrees with the known area law relation in QFTs.

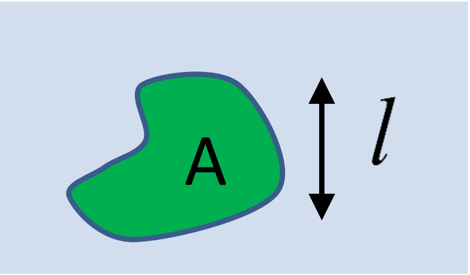
Relation to BH Entropy

We can regard the HEE as a generalization of BH entropy.



AdS BH = CFT at finite temp. ⇒Mixed state: $S_A \neq S_B$!

General Behavior of HEE (=EE in CFT$_{d+1}$) [Ryu−TT 06,⋯]



$$S_A = \frac{\pi^{d/2} R^d}{2G_N^{(d+2)}\Gamma(d/2)}\left[p_1 \left(\frac{l}{\varepsilon}\right)^{d-1} + p_3 \left(\frac{l}{\varepsilon}\right)^{d-3} + \cdots \right.$$

$$\left. \cdots + \begin{cases} p_{d-1}\left(\frac{l}{\varepsilon}\right) + p_d & (\text{if } d+1 = \text{odd}) \\ p_{d-2}\left(\frac{l}{\varepsilon}\right)^2 + q \log\left(\frac{l}{\varepsilon}\right) & (\text{if } d+1 = \text{even}) \end{cases} \right],$$

where $p_1 = (d-1)^{-1},\ p_3 = -(d-2)/[2(d-3)], \ldots.$

..... $q = (-1)^{(d-1)/2}(d-2)!!/(d-1)!!$.

Area law divergence

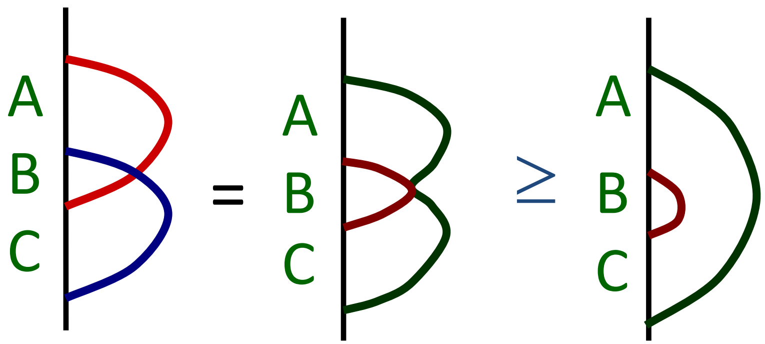
A universal quantity (**F**) which characterizes **odd dim. CFT**.

Agrees with **conformal anomaly** (central charge) in **even dim. CFT**

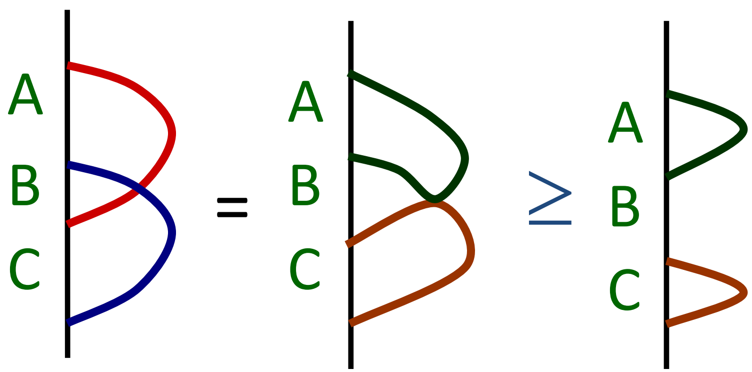
Algebraic properties in Quantum Information
⇔ **Geometric properties** in Gravity

Holographic Proof of Strong Subadditivity(SSA)

[Headrick−TT 07]



$$\Rightarrow S_{AB} + S_{BC} \geq S_{ABC} + S_B$$



$$\Rightarrow S_{AB} + S_{BC} \geq S_A + S_C$$

$$(\text{Note}: \quad AB \equiv A \cup B)$$

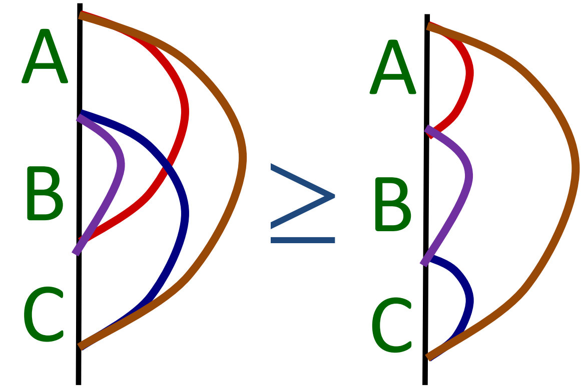
"Triangle inequalities in Geometry = SSA"

Monogamy of Mutual Information [Hayden-Headrick-Maloney 11]

The holographic mutual information

$$I(A:B) = S_A + S_B - S_{AB}$$

has a special property called ***monogamy***.



$$I(A:BC) \geq I(A:B) + I(A:C)$$

$$\Leftrightarrow I_3(A,B,C) \equiv S_A + S_B + S_C + S_{ABC} - S_{AB} - S_{BC} - S_{AC} \leq 0$$

Comments:

- This property is special to holographic CFTs.

  [cf. For massive free fermion QFT: $I_3 > 0$ Casini-Fosco-Huerta 05]

- This property also leads to the *Cadney-Linden-Winter inequality* as well as *strong superadditivity* of Hol. MI.

(1-4) HEE for Time-dependent Backgrounds

[Hubeny–Rangamani–TT 07]

A generic Lorentzian asymptotic AdS spacetime is dual to a time dependent state $|\Psi(t)\rangle$ in the dual CFT.

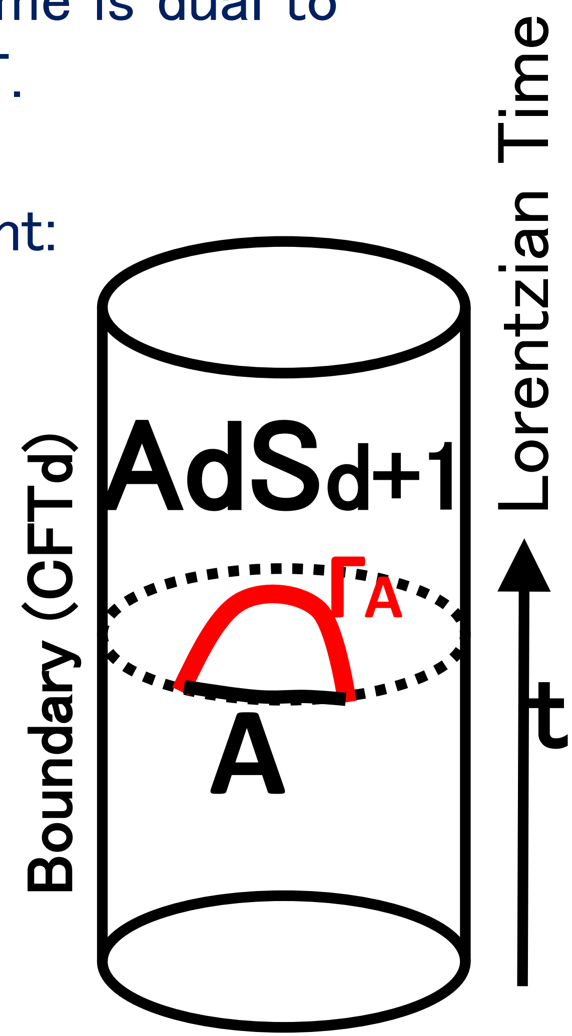
The entanglement entropy gets time–dependent:

$$\rho_A(t) = \mathrm{Tr}_B[|\Psi(t)\rangle\langle\Psi(t)|] \Rightarrow S_A(t).$$

This is computed by the holographic formula:

$$S_A(t) = \mathrm{Min}_{\Gamma_A}\mathrm{Ext}_{\Gamma_A}\left[\frac{A(\Gamma_A)}{4G_N}\right]$$

$$\partial A = \partial\gamma_A \text{ and } A \sim \gamma_A \, .$$

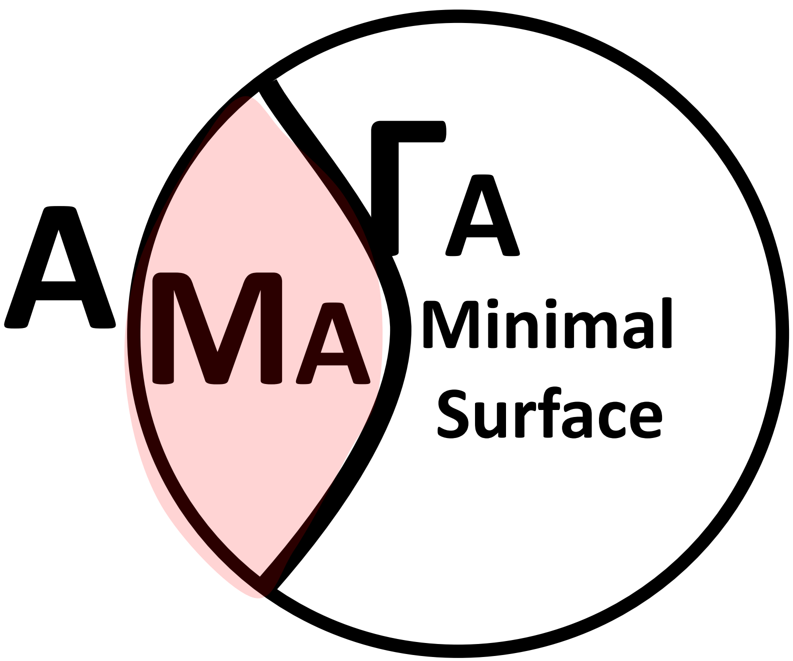


Entanglement Wedges

Which bulk region is dual to a given region A in CFT ?

⇒ Entanglement Wedge M_A (note: we took a time slice)

M_A = A region surrounded by A and Γ_A (on time slice)



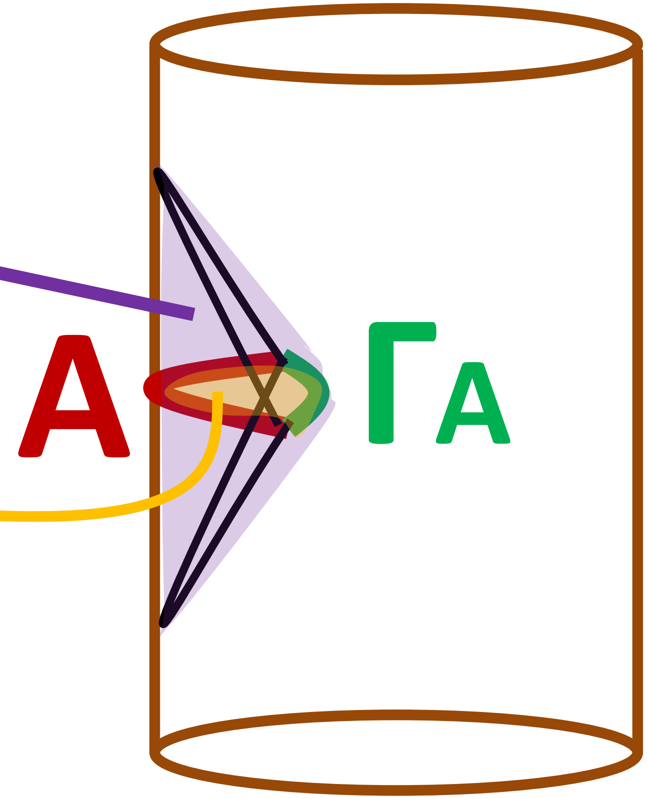
ρ_A in CFT

$\Leftrightarrow \rho_{MA}$ in AdS gravity

[Hamilton−Kabat−Lifschytz−Lowe 2006, Czech−Karczmarek−Nogueira−Raamsdonk 2012, Wall 2012, Headrick−Hubeny−Lawrence−Rangamani 2014, Jafferis−Lewkowycz−Maldacena−Suh 2015, Dong−Harlow−Wall 2016, . . .]

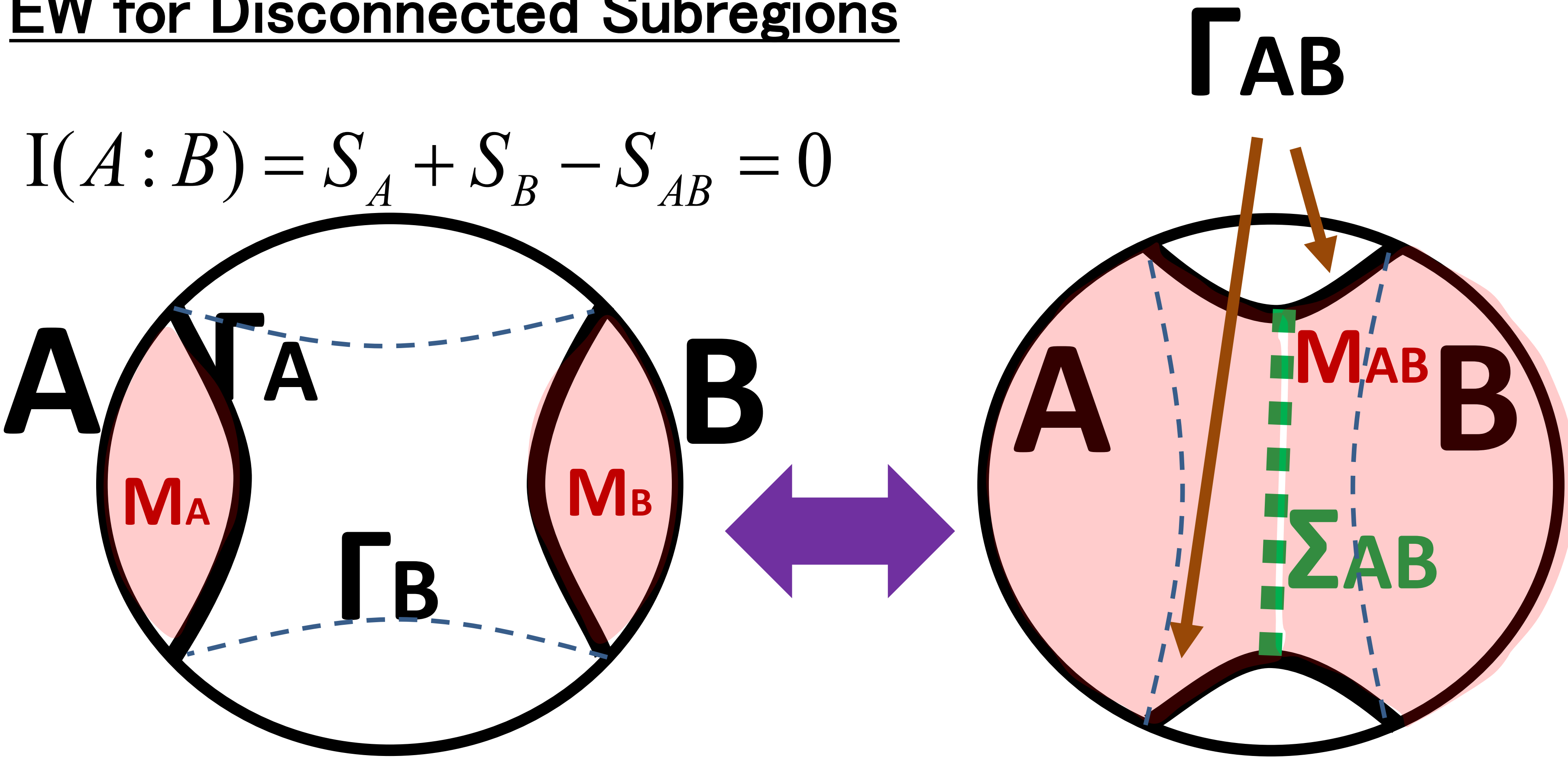
Covariant Definition of EW

Entanglement Wedge
=Domain of dependence
of M_A



EW for Disconnected Subregions

$$\mathrm{I}(A:B) = S_A + S_B - S_{AB} = 0$$



$M_{AB} = M_A \cup M_B$, $\Gamma_{AB} = \Gamma_A \cup \Gamma_B$

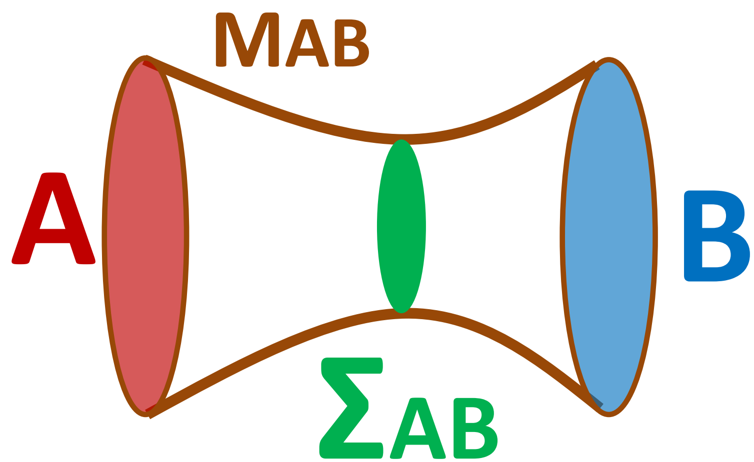
$$\mathrm{I}(A:B) > 0$$

Σ_{AB}= Minimal Surface which divides M_{AB} into A side and B side

Entanglement Wedge Cross Section (EWCS)

We define a quantity called ***EW cross section*** by

$$E_W(\rho_{AB}) = \frac{\text{Area}(\Sigma_{AB})}{4G_N}$$



Gravity dual conjecture of EWCS

$$E_W(\rho_{AB}) = E_P(\rho_{AB})$$

Entanglement of Purification

[Terhal-Horodecki-Leung-Divincenzo quant-ph/0202044]

Note: When ρ_{AB} is a pure state, we simply have

$$E_W(\rho_{AB}) = E_P(\rho_{AB}) = S_A = S_B .$$

Definition of Entanglement of Purification (EoP)

First let us explain the purification procedure:

A given density matrix for H_C : $\rho_C = \sum_i \lambda_i |i\rangle_C \langle i|$.

We can always describe this state as a pure state by extending the Hilbert space:

$$H_C \to H_C \otimes H_D \qquad |\Psi\rangle_{CD} = \sum_i \sqrt{\lambda_i}\, |i\rangle_C |i\rangle_D$$

such that $\rho_C = \mathrm{Tr}_D \left[|\Psi\rangle\langle\Psi| \right]$.

Note: there are infinite many ways to do this.

Consider all purifications $|\Psi\rangle_{A\tilde{A}B\tilde{B}}$ of ρ_{AB} in the extended Hilbert space: $H_A \otimes H_B \rightarrow H_A \otimes H_B \otimes H_{\tilde{A}} \otimes H_{\tilde{B}}$.

Then, **Entanglement of Purification (EoP)** is defined by

$$\mathrm{E_P}(\rho_{AB}) = \underset{\text{All purifications } |\Psi\rangle \text{ of } \rho_{\mathrm{AB}}}{\mathrm{Min}} S_{A\tilde{A}}\left(|\Psi\rangle_{A\tilde{A}B\tilde{B}}\right)$$

$$\rho_{AB} = \mathrm{Tr}_{\tilde{A}\tilde{B}}\left[|\Psi\rangle\langle\Psi|\right].$$

Entanglement Entropy

Note: $\mathrm{E_p}(\rho_{AB}) \geq 0$ and $\mathrm{E_p}(\rho_{AB}) = 0 \Leftrightarrow \rho_{AB} = \rho_A \otimes \rho_B$.

Other Conjectures of EWCS

[Dutta-Faulkner 2019]

Proposal 2: Reflected Entropy

$$2\cdot \mathrm{E_W}(\rho_{\mathrm{AB}}) = \mathrm{S_R}(\rho_{\mathrm{AB}})$$

$$\mathrm{S_R}(\rho_{AB}) \equiv S_{A\tilde{A}}\left(|\Psi\rangle_{A\tilde{A}B\tilde{B}}\right), \quad \text{s.t.} \quad |\Psi\rangle_{A\tilde{A}B\tilde{B}} = \sum_{i,j}\left(\sqrt{\rho_{AB}}\right)_{ij}|i\rangle_{AB}|j\rangle_{\tilde{A}\tilde{B}}$$

Proposal 3: Odd Entropy

$$\mathrm{E_W}(\rho_{\mathrm{AB}}) = \mathrm{S_{odd}}(\rho_{\mathrm{AB}})$$

[Tamaoka 2018]

Partial transposition for B

$$\mathrm{S_{odd}}(\rho_{AB}) \equiv \lim_{n_{odd}\to 1}\frac{1}{1-n_{odd}}\log\left(\rho_{AB}^{T_B}\right)^{n_{odd}}. \quad \left(\rho_{AB}^{T_B}\right)_{ab,AB} \equiv \left(\rho_{AB}\right)_{aB,Ab}$$

Proposal 4: Logarithmic negativity

[KudlerFlam-Ryu 2018]

$$\mathrm{LN}(\rho_{AB}) \equiv \log\sqrt{\left(\rho_{AB}^{T_B}\right)^2} \quad .$$

$$\frac{3}{2}\cdot \mathrm{E_W}(\rho_{\mathrm{AB}}) = \mathrm{LN}(\rho_{\mathrm{AB}})$$

(1-5) Holographic Pseudo Entropy

Question: Ver 3. Holographic Entropy Formula ?

Minimal areas in ***Euclidean time dependent*** asymptotically AdS spaces

= What kind of QI quantity (Entropy ?) in CFT ?

➡ **The answer is Pseudo Entropy !**

[Nakata-Taki-Tamaoka-Wei-TT, 2020]

Definition of Pseudo (Renyi) Entropy

Consider two quantum states $|\psi\rangle$ and $|\varphi\rangle$, and define the *transition matrix*:

$$\tau^{\psi|\varphi} = \frac{|\psi\rangle\langle\varphi|}{\langle\varphi|\psi\rangle}.$$

We decompose the Hilbert space as $H_{tot} = H_A \otimes H_B$.

and introduce the reduced transition matrix:

$$\tau_A^{\psi|\varphi} = \mathrm{Tr}_B\left[\tau^{\psi|\varphi}\right]$$

Pseudo Entropy

$$S\left(\tau_A^{\psi|\varphi}\right) = -\mathrm{Tr}\left[\tau_A^{\psi|\varphi}\log\tau_A^{\psi|\varphi}\right].$$

Renyi Pseudo Entropy

$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = \frac{1}{1-n}\log\mathrm{Tr}\left[\left(\tau_A^{\psi|\varphi}\right)^n\right].$$

Basic Properties of Pseudo Entropy (PE)

- In general, $\tau_A^{\psi|\varphi}$ is not Hermitian. Thus PE is complex valued.

- If either $|\psi\rangle$ or $|\varphi\rangle$ has no entanglement (i.e. direct product state) , then
$$S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = 0.$$

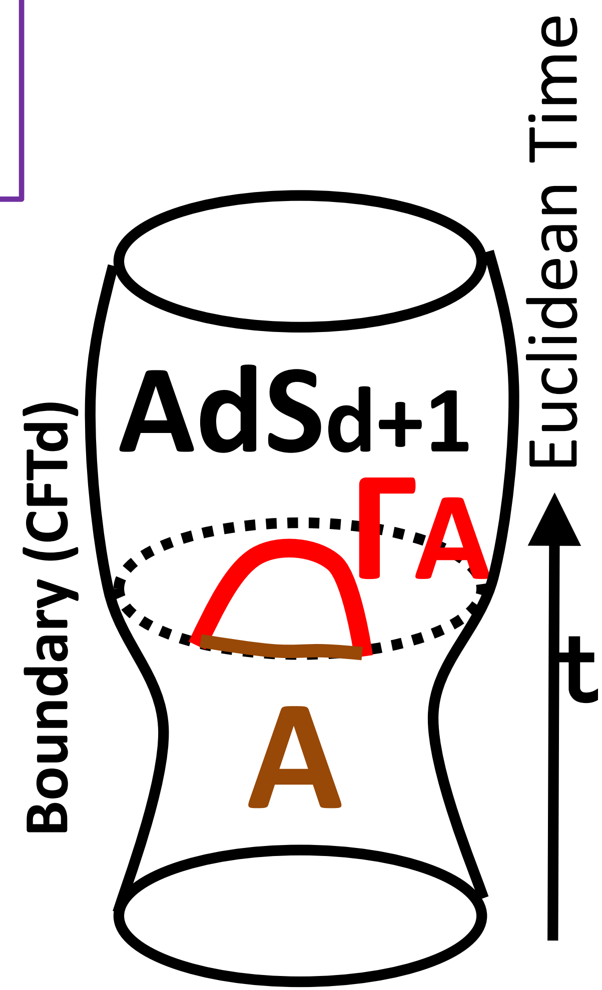
- We can show $S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = \left[S^{(n)}\left(\tau_A^{\varphi|\psi}\right)\right]^{\dagger}$.

- We can show $S^{(n)}\left(\tau_A^{\psi|\varphi}\right) = S^{(n)}\left(\tau_B^{\psi|\varphi}\right)$. →"$S_A=S_B$"

- If $|\psi\rangle=|\varphi\rangle$, then $S^{(n)}\left(\tau_A^{\psi|\varphi}\right)$ = Renyi entropy.

Holographic Pseudo Entropy

$$S\left(\tau_A^{\psi|\varphi}\right) = \mathrm{Min}_{\Gamma_A}\left[\frac{A(\Gamma_A)}{4G_N}\right]$$



Basic Properties

(i) If ρ_A $is\ pure$, $S\left(\tau_A^{\psi|\varphi}\right)$ =0.

(ii) If ψ or φ $is\ not\ entangled,$

$S\left(\tau_A^{\psi|\varphi}\right)$ =0.

→This follows from AdS/BCFT [TT 2011]

(iii) $S\left(\tau_A^{\psi|\varphi}\right) = S\left(\tau_B^{\psi|\varphi}\right)$. **"SA=SB"**

Pseudo Entropy and Quantum Phase Transitions

[Mollabashi–Shiba–Tamaoka–Wei–TT 20, 21]

Basic Properties of Pseudo entropy in QFTs

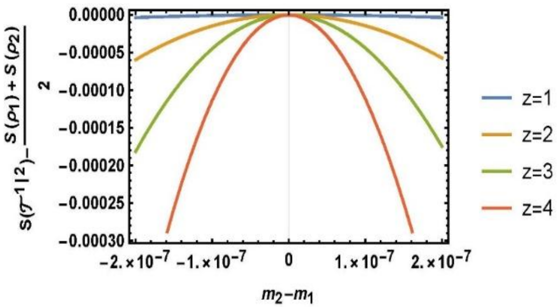
[1] Area law

$$S_A \sim \frac{\mathrm{Area}(\partial A)}{\varepsilon^{d-1}} + (\text{subleading terms}),$$

[2] The difference

$$\Delta S = S\left(\tau_A^{1|2}\right) + S\left(\tau_A^{1|2}\right) - S(\rho_A^1) - S(\rho_A^2)$$

is negative if $|\psi_1\rangle$ and $|\psi_2\rangle$ are in a same phase.



PE in a 2 dim. free scalar when we change its mass.

➡ **What happen if they belong to different phases ?**
Can ΔS be positive ?

Quantum Ising Chain with a transverse magnetic field

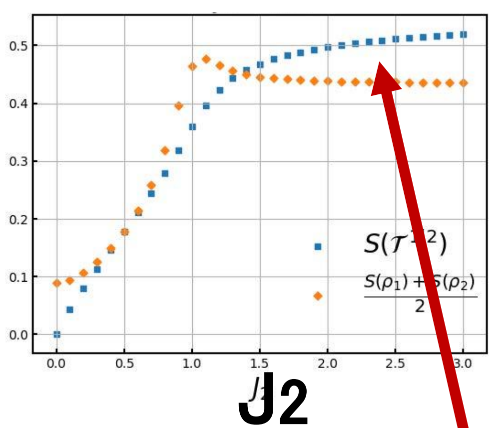
$$H = -J\sum_{i=0}^{N-1}\sigma_i^z\sigma_{i+1}^z - h\sum_{i=0}^{N-1}\sigma_i^x,$$

Ψ1→ vacuum of H(J1)
Ψ2→ vacuum of H(J2)
(We always set h=1)

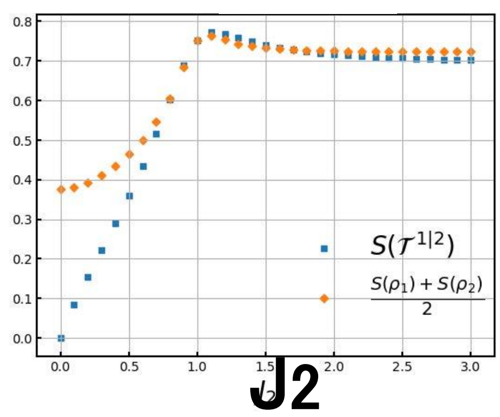
J<1 Paramagnetic Phase
J>1 Ferromagnetic Phase

N=16, NA=8

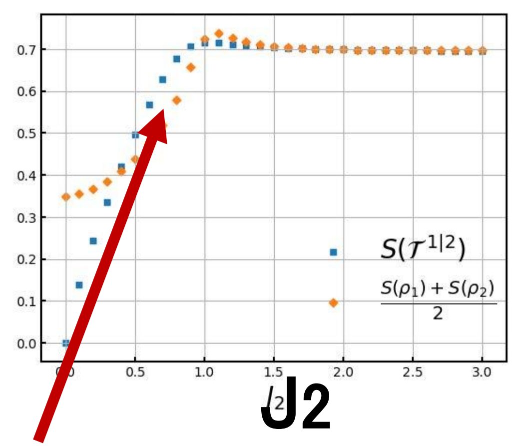
J1=1/2



J1=1



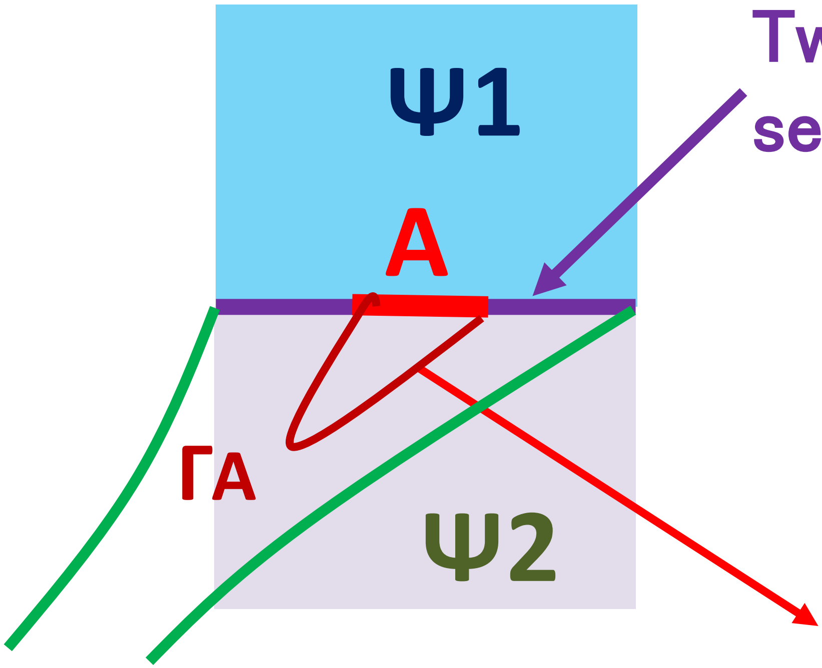
J1=2



$$\Delta S = S\left(\tau_A^{1|2}\right) + S\left(\tau_A^{1|2}\right) - S(\rho_A^1) - S(\rho_A^2) > 0$$

We find
when Ψ1 and Ψ2 are in different phases !

Heuristic Interpretation



Two gapped phases are separated by a gapless phase.
CFT !

PE is enhanced ! $\Delta S > 0$

The gapless interface (edge state) also occurs in topological orders.
➔Topological pseudo entropy [Nishioka-Taki-TT 2021].

② AdS/BCFT

(2-1) BCFT (Boundary Conformal Field Theory)

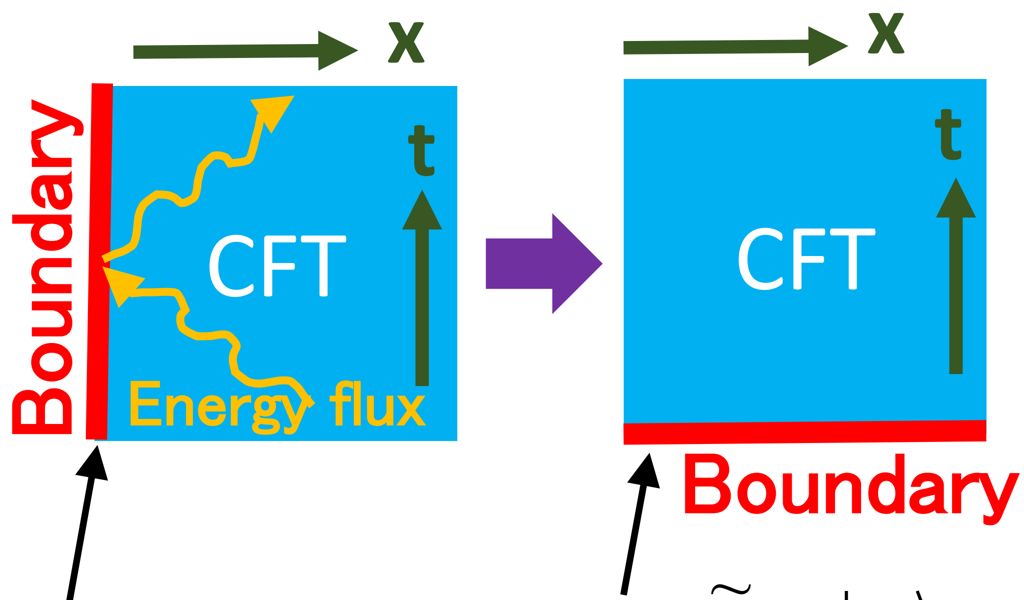
For special boundary conditions, a part of conformal symmetries are preserved, called the boundary conformal field theory (BCFT).

[Cardy 1984, .., McAvity–Osborn 1995, ··· ; Cond-mat application: Kondo effect]

d dim CFT : SO(2,d)

U

d dim. BCFT: SO(2,d−1)

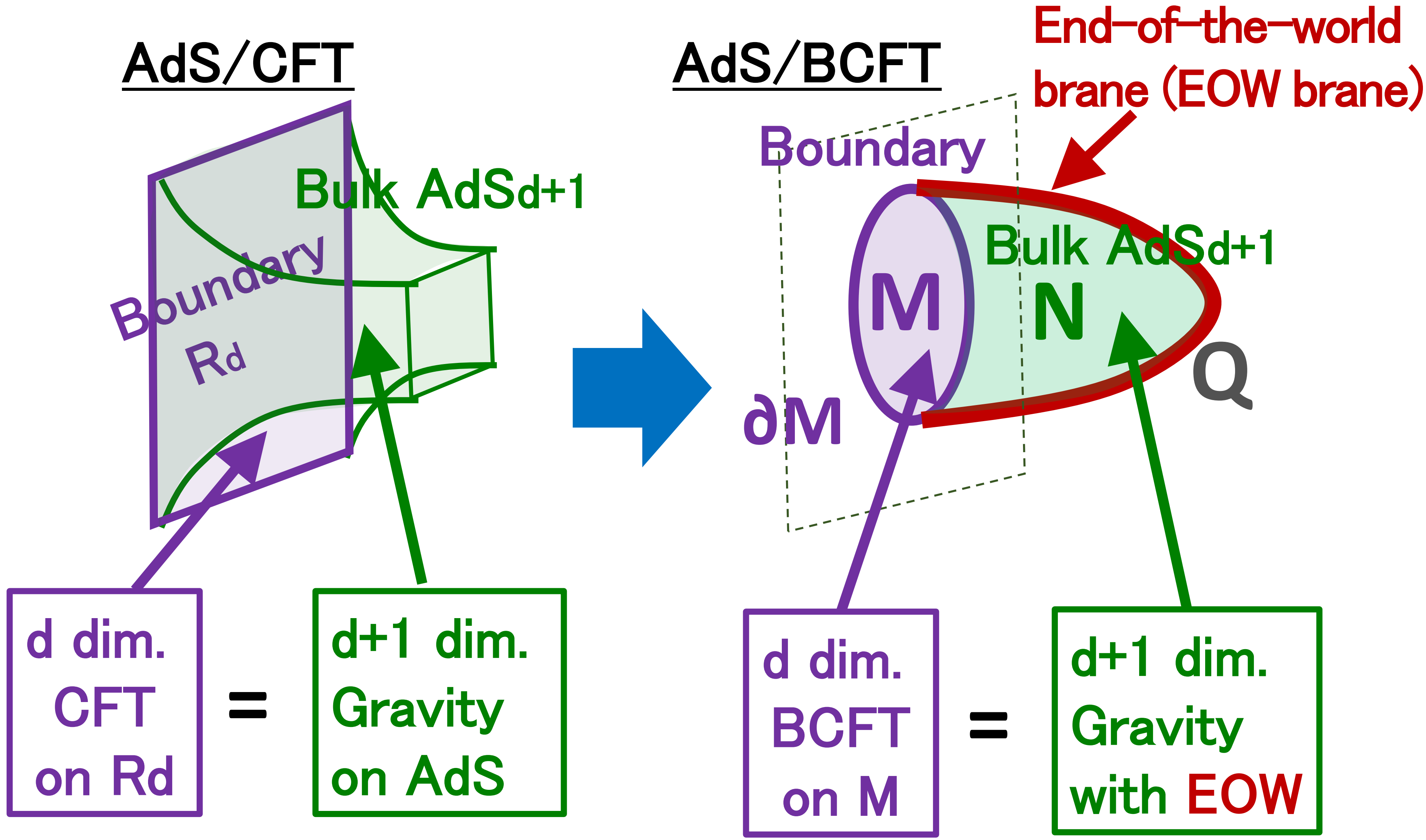


$$T_{tx} \propto \left[T(z) - \bar{T}(\bar{z})\right]_{Bdy} = 0$$

$$(L_n - \tilde{L}_{-n})\,|B\rangle = 0$$

Boundary state

(2-2) AdS/BCFT



AdS/CFT
Bulk AdSd+1
Boundary
Rd
d dim.
CFT
on Rd
=
d+1 dim.
Gravity
on AdS
AdS/BCFT
End-of-the-world
brane (EOW brane)
Boundary
Bulk AdSd+1
M
N
Q
∂M
d dim.
BCFT
on M
=
d+1 dim.
Gravity
with EOW

The gravity action in Euclidean signature looks like

Gibbons -Hawking term

$$I = \frac{1}{16\pi G_N}\int_N \sqrt{-g}(R - 2\Lambda + L_{matter}) + \frac{1}{8\pi G_N}\int_Q \sqrt{-h}(K + L^Q_{matter}).$$

Bulk matter fields

Matter fields localized on Q

The coordinate of Q and its induced metric are x^a and h^{ab} .

We define the extrinsic curvature and its trace

$K_{ab} = \nabla_a n_b, \quad K = h^{ab} K_{ab}$. (n^a is a unit vector normal to Q.)

e.g. Gaussian normal coordinate: $ds^2 = d\rho^2 + h_{ab}(\rho, x) dx^a dx^b$

➡ $$K_{ab} = \frac{1}{2}\partial_\rho h_{ab}(\rho, x).$$

Variation:
$$\delta I = \frac{1}{16\pi G_N}\int_Q \sqrt{-h}\,(K_{ab} - Kh_{ab} - T^Q_{ab})\,\delta h^{ab}.$$

At the AdS boundary M, we impose the Dirichlet boundary condition $\delta h^{ab} = 0$ following the standard AdS/CFT argument.

On the other hand, at the new boundary Q, we argue to require the Neumann b.c. :

$$\boxed{K_{ab} - Kh_{ab} - T^Q_{ab} = 0}$$ `boundary Einstein eq.'

Why Neumann b.c. (brane-world type) ?

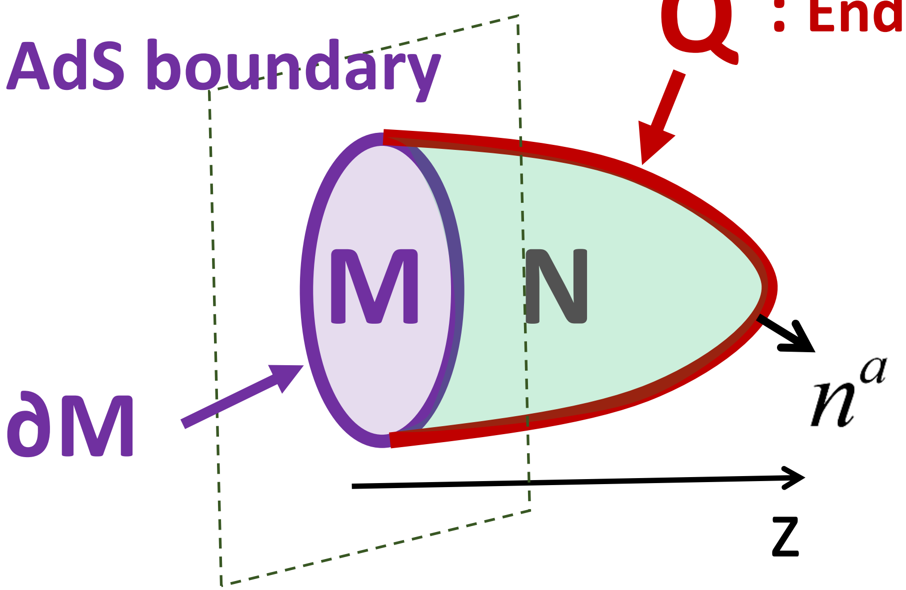
(1) Keep the boundary dynamical. New data at Q should not be required.

(2) Orientifolds in string theory lead to this condition.

In general, this AdS/BCFT description is a hard wall approximation.

Summary of AdS/BCFT construction

CFT on a manifold M with a boundary ∂M = Gravity on an asymptotically AdS space N, s.t. ∂N=M∪Q



Q : End of the world (EOW) brane

We impose Neumann b.c.:

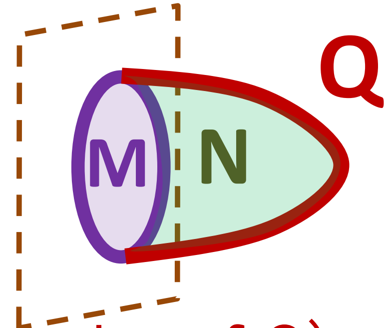
$$K_{ab} - K h_{ab} - T^{Q}_{ab} = 0$$

Depend on types of EOW brane.

Extrinsic curvature:

$$K_{ab} = \nabla_a n_b, \quad K = h^{ab} K_{ab}$$

Holographic Dual of BCFT



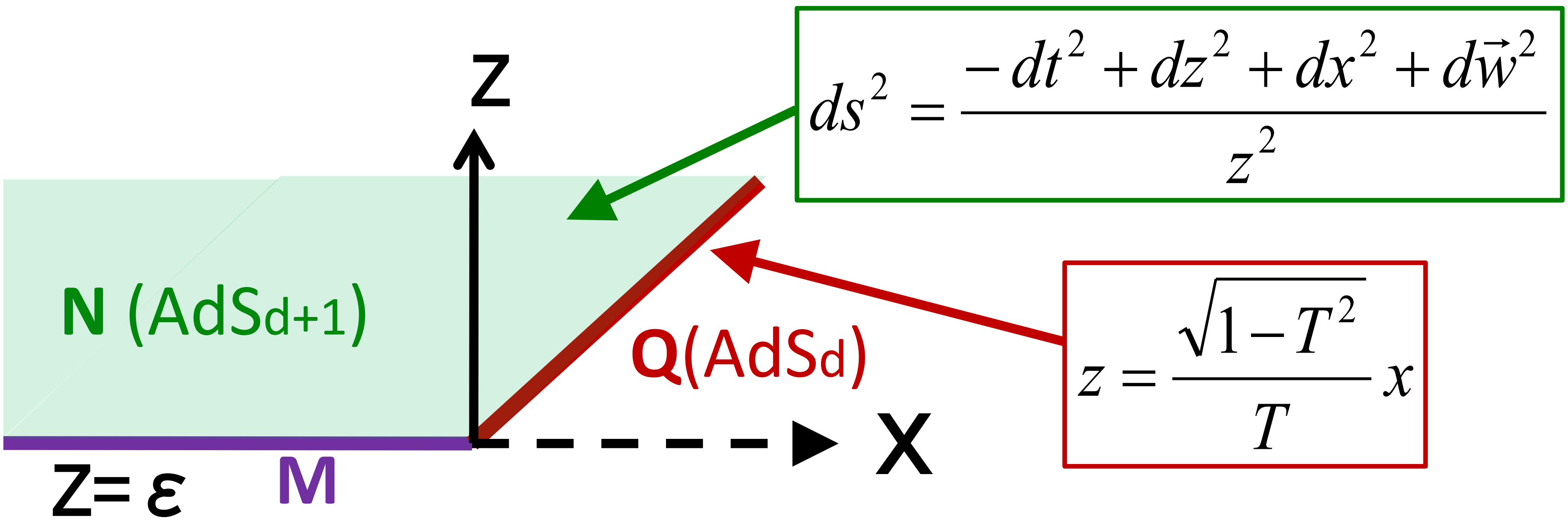
To preserve the BCFT symmetry, we choose

$$T^{Q}_{ab} \propto h_{ab} \;\Rightarrow\; T^{Q}_{ab} = -T\, h_{ab}$$ (T is the tension of Q).

The Neumann b.c. looks like

$$K_{ab} = (K - T)\, h_{ab}\,.$$

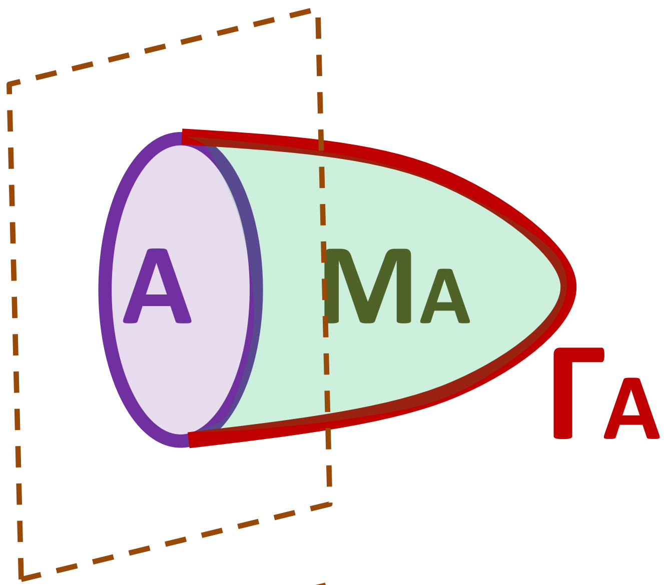
Example: Dual of BCFT on a half space



$$ds^2 = \frac{-dt^2 + dz^2 + dx^2 + d\vec{w}^2}{z^2}$$

$$z = \frac{\sqrt{1-T^2}}{T}\, x$$

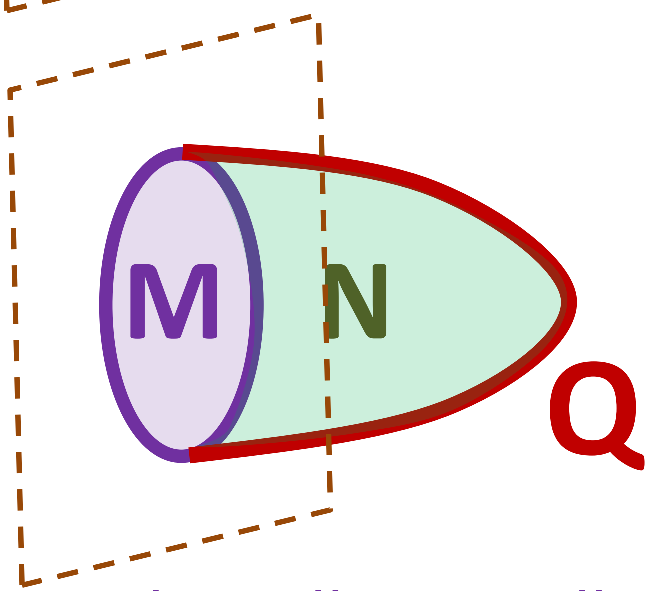
Differences between two “subregion/subregion duality”



[1] Entanglement Wedge

⇒ ΓA is extremal surface.
(no back-reactions)

$$h^{ab}K_{ab}=0$$



[2] AdS/BCFT

⇒ Q is totally geodesic surface
or it generalizations.

$$K_{ab}=\text{fixed}$$

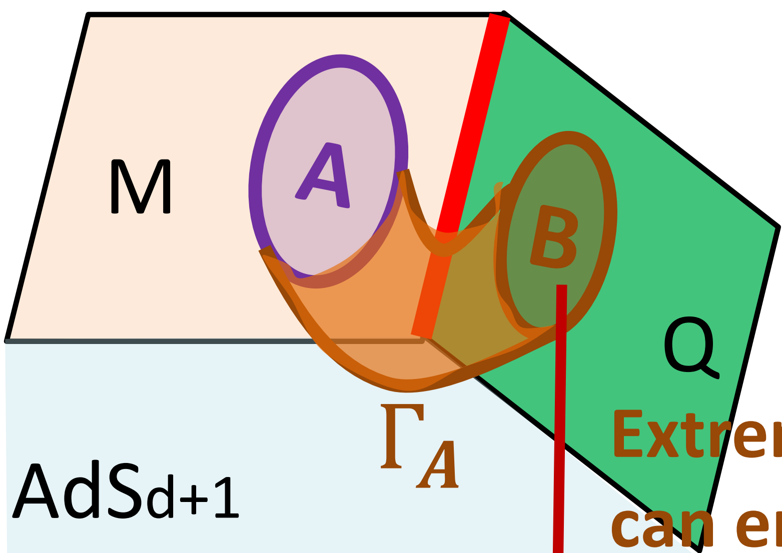
⇒ Surface Q back-reacts !

In this talk, we will see interesting interplay between them.

(2-3) HEE in AdS/BCFT

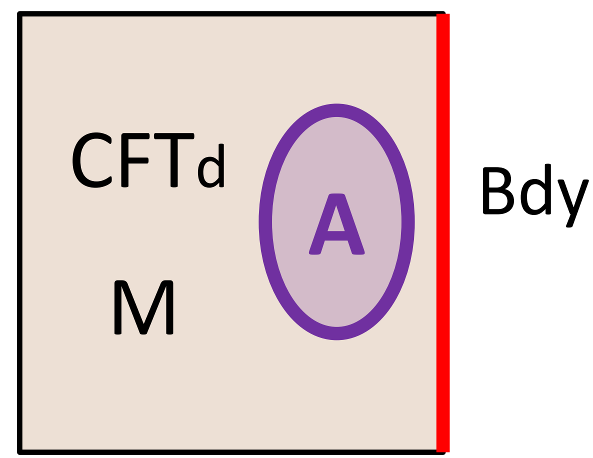
[TT 2011, Fujita-Tonni-TT 2011]

$$S_A = \underset{\Gamma_A, B}{\text{Min Ext}} \left[\frac{\text{Area}(\Gamma_A)}{4G_N} \right] \qquad \partial\Gamma_A = \partial A \cup \partial B$$



AdS/BCFT

=



Extremal Surfaces can end on Q !

This region B is now known as an **Island** !

$$\textbf{Island formula:} \quad S_A = \text{Min} \left[\frac{\text{Area}(\Sigma)}{4G_N} + S_{A\cup\Sigma} \right]$$

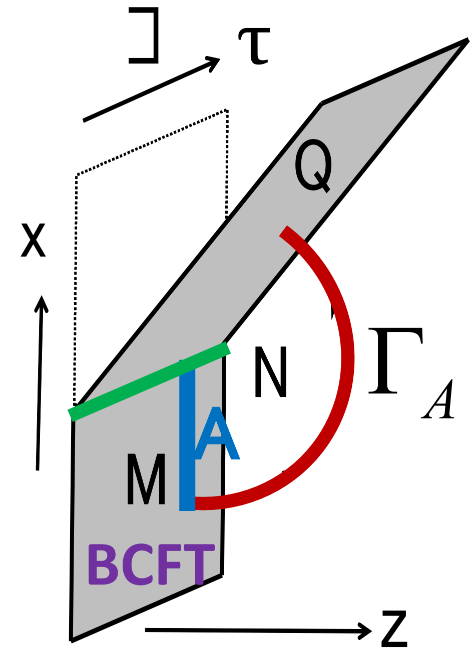
HEE in AdS3/BCFT2

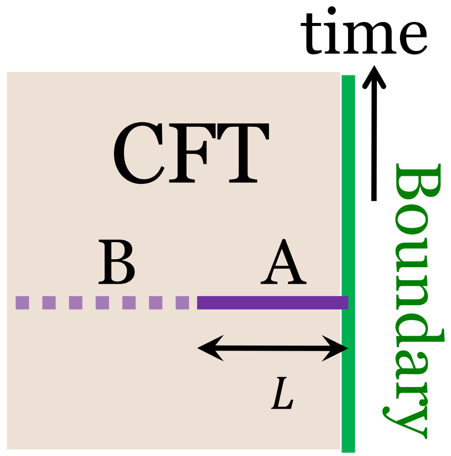
The holographic EE is obtained as

$$S_A = \frac{\text{Length}}{4G_N} = \frac{c}{6}\log\frac{2L}{\varepsilon} + \frac{c}{12}\log\frac{1+T}{1-T} \ .$$

cf. CFT Result

$$S_A = \underbrace{\frac{c}{6}\log\frac{2L}{\varepsilon}}_{\text{Bulk Part}} \quad + \quad \underbrace{\log g}_{\text{Boundary Entropy (g-function)}} \ .$$



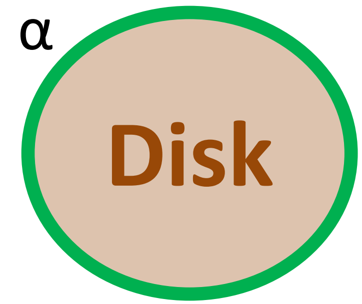


(2-4) Holographic g-theorem

Definitions of g-function (boundary Entropy) [Affleck-Ludwig 1991]

Def 1 (Disk Amplitude)

$$S_{bdy(\alpha)} = \log g_\alpha \,, \qquad g_\alpha \equiv \langle 0 | B_\alpha \rangle.$$



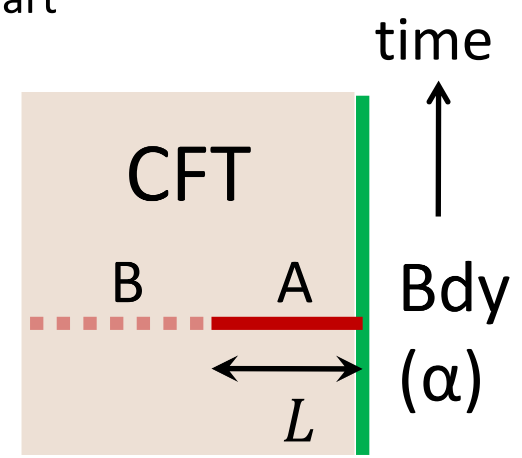
Def 2 (Cylinder Amplitude)

$$Z^{cylinder}_{(\alpha,\beta)} = \langle B_\alpha | e^{-HL} | B_\beta \rangle \underset{L\to\infty}{\approx} \underbrace{g_\alpha g_\beta}_{\text{Boundary Part}} \underbrace{e^{-E_0 L}}_{\text{Bulk Part}}.$$

Def 3 (Entanglement Entropy)

In 2d BCFT, the EE behaves like

$$S_A = \underbrace{\frac{c}{6}\log\frac{2L}{\varepsilon}}_{\text{Bulk Part}} + \underbrace{\log g_\alpha}_{\text{Boundary Part}} \quad .$$



[Calabrese-Cardy 2004]

Derivation of Holographic g-Theorem

Consider the surface Q defined by $x = x(z)$ in the Poincare metric

$$ds^2 = R^2\left(\frac{dz^2 - dt^2 + dx^2 + (d\vec{w})^2}{z^2}\right).$$

We impose the null energy condition for the boundary matter

i.e. $T^Q_{ab} N^a N^b \geq 0$ for any null vector N^a.

[cf. Hol. C-theorem: Freedman-Gubser-Pilch-Warner 1999, Myers-Sinha 2010]

For the null vector, $N^t = 1,\ \ N^z = 1/\sqrt{1+(x')^2}\ ,\ N^x = x'/\sqrt{1+(x')^2}\ ,$

we find the constraint

$$(K_{ab} - Kh_{ab})N^a N^b = -\frac{R \cdot x''}{z(1+(x')^2)^{3/2}} \geq 0.$$

Thus we simply get $x''(z) \le 0$ from the null energy condition.

Let us define the holographic g-function by

$$\log g(z) = \frac{R^{d-1}}{4G_N} \cdot \mathrm{Arcsinh}\left(\frac{x(z)}{z}\right) = \frac{R^{d-2}}{4G_N} \cdot \rho_*(z).$$

Then it is easy to see $\dfrac{\partial \log g(z)}{\partial z} = \dfrac{x'(z)z - x(z)}{\sqrt{z^2 + x(z)^2}} \le 0$,

because $(x'z - x)' = x''z \le 0$.

For d=2, at fixed points $\log g(z)$ agrees with the boundary entropy.

For any dimension d, we find that $\rho_*(z)$ is a monotonically decreasing function of the length scale z.

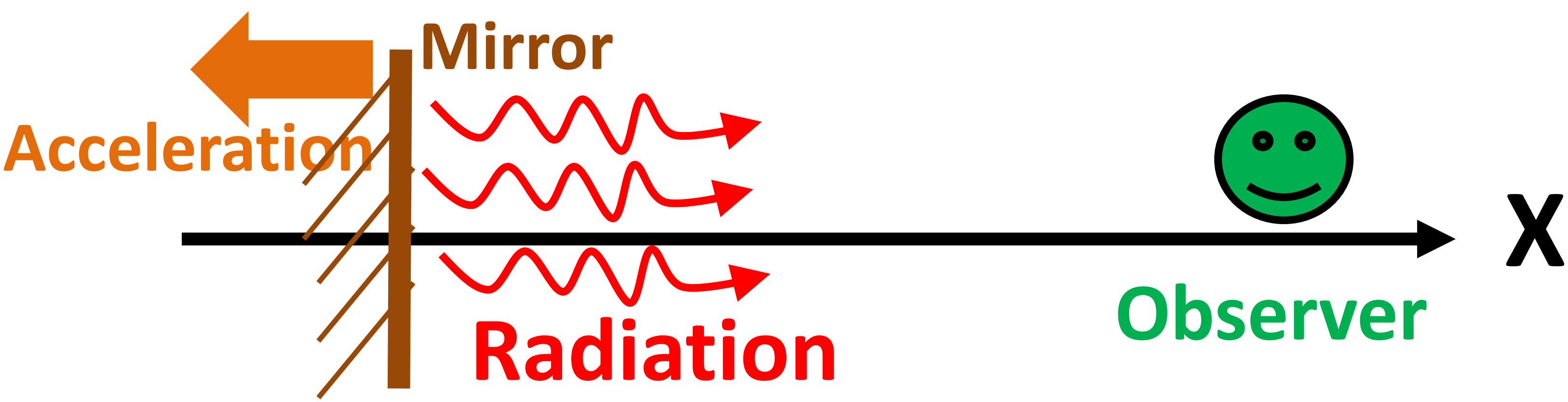
➡ This is our holographic g-theorem !

③ Moving Mirror and EE

Moving mirror

Moving mirrors have been known for a while as instructive models which mimic the Hawking radiation from Black holes.

[see e.g. Birrell-Davies text book]

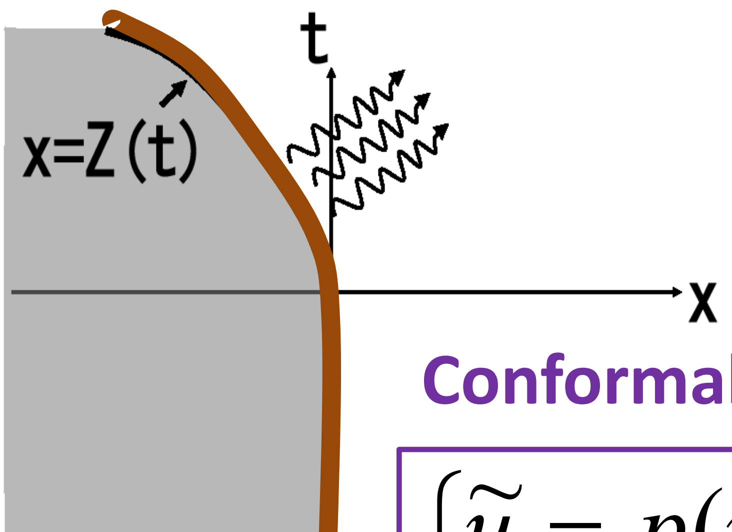


This provides an interesting class of non-equilibrium processes, where quantum entanglement gets crucial. [cf. quantum quenches]

(3-1) BCFT Description

We focus on two dim. CFTs. Then we can apply conformal mapping to solve the moving mirror problem.
We write a mirror trajectory as x=Z(t).

Moving Mirror



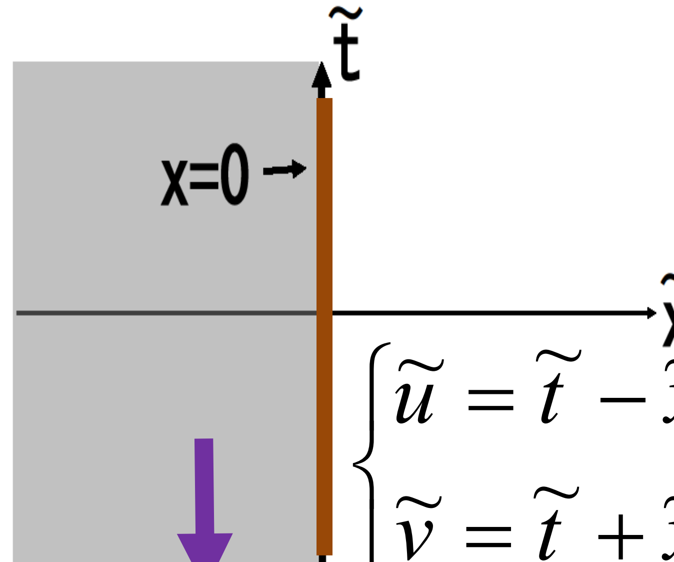
$$\begin{cases} u = t - x \\ v = t + x \end{cases}$$

Conformal Map

$$\begin{cases} \tilde{u} = p(u) \\ \tilde{v} = v \end{cases}$$

In coming vacuum

Static Mirror



$$\begin{cases} \tilde{u} = \tilde{t} - \tilde{x} \\ \tilde{v} = \tilde{t} + \tilde{x} \end{cases}$$

Standard Setup of BCFT

Boundary Conformal Field Theory

i.e. Conformal symmetry is partially preserved on the bdy.

Example 1 : Escaping Mirror (Constant Radiation)

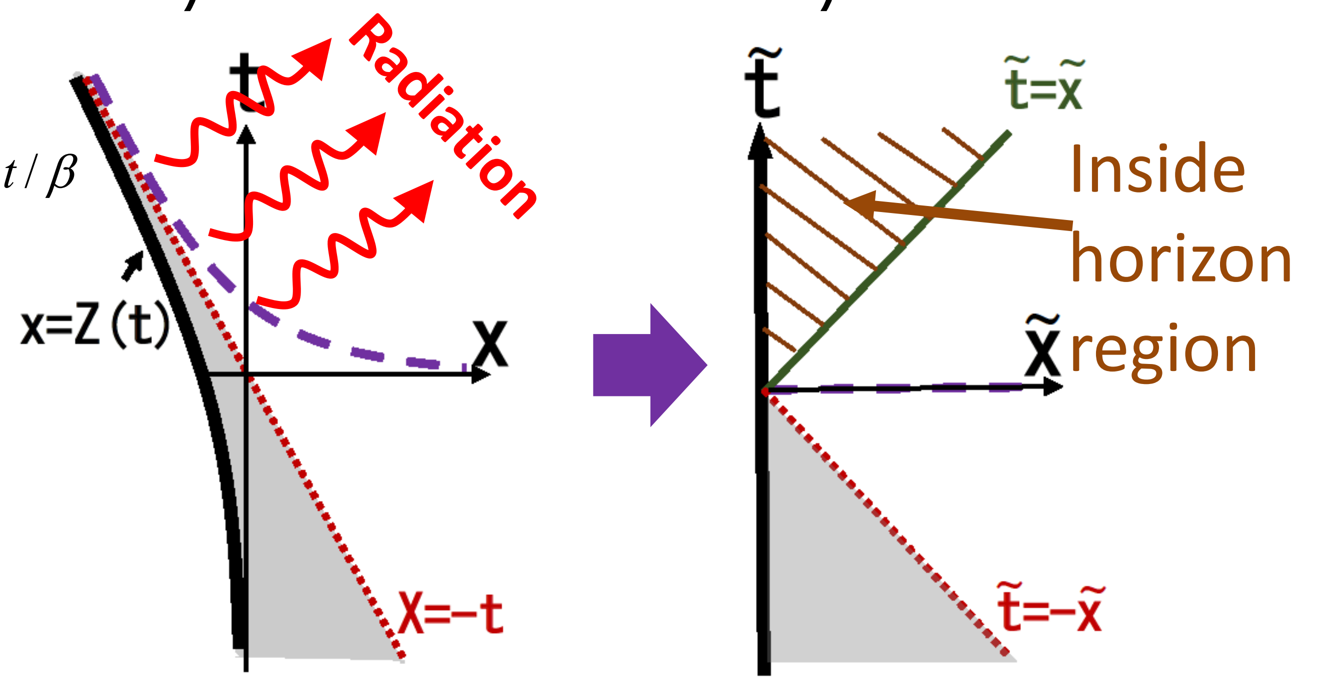
$$p(u) = -\beta \log(1 + e^{-u/\beta})$$

Energy flux : $T_{uu} = \dfrac{c}{24\pi}\left(\dfrac{3}{2}\dfrac{(P'')^2}{(P')^2} - \dfrac{p'''}{p'}\right)$

$$= \frac{c}{48\pi\beta^2}\left(1 - \frac{1}{\left(1+e^{u/\beta}\right)^2}\right) \approx \frac{c}{48\pi\beta^2}$$

Thermal flux at temperature T=1/β

$$z(t) \underset{t\to\infty}{\approx} -t - \beta\, e^{-2t/\beta}$$

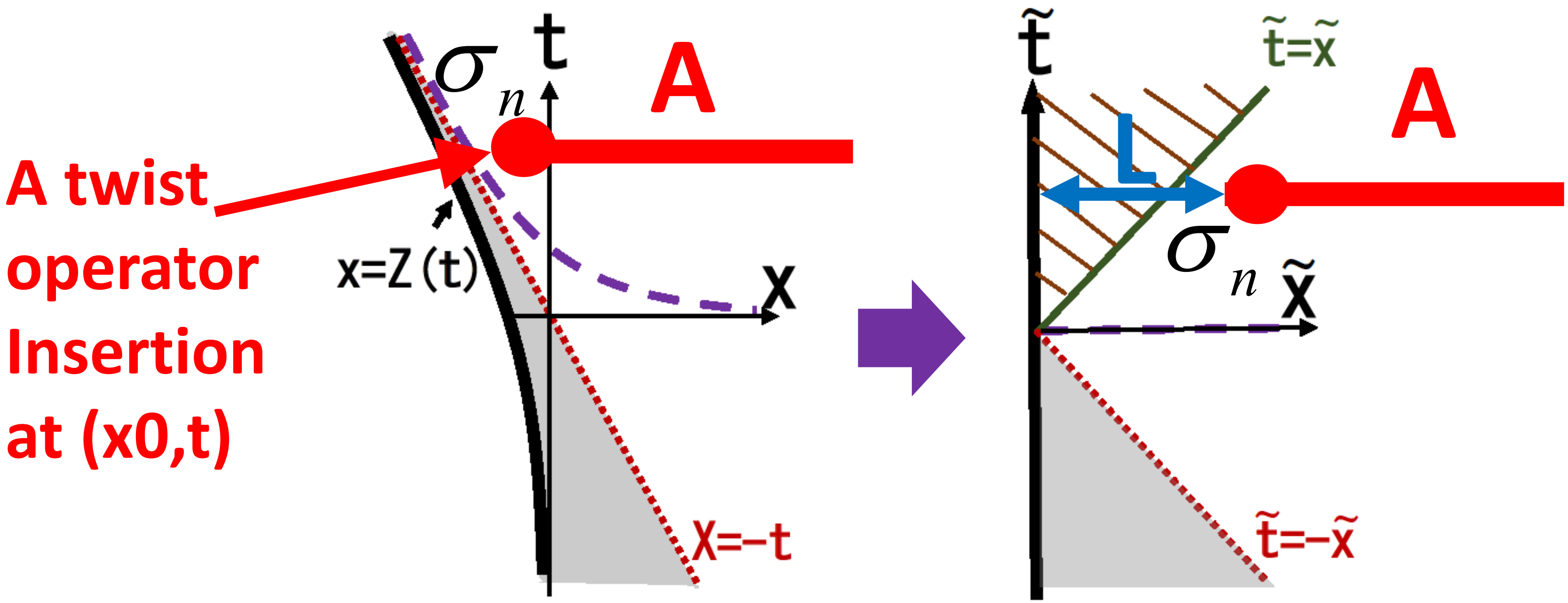


(3-2) Computing Entanglement Entropy

Calculation of Entanglement Entropy (EE)

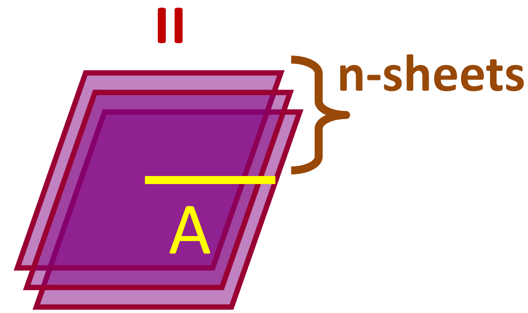
To get a universal result, we choose the subsystem A to be a semi-infinite line A=[x0,∞] at time t.

We consider the EE between A and its compliment.



We can calculate the EE via the replica method.

$$\mathrm{Tr}[(\rho_A)^n]$$



$$\mathrm{Tr}\left[(\rho_A)^n\right] = \langle \sigma_n \rangle = \frac{g}{L^{\Delta_n}}, \quad \Delta_n = \frac{c}{12}(n - 1/n).$$

where $g = e^{S_{bdy}}$ is the g-function or bdy entropy.
[Affleck-Ludwig 1991]

By applying the conformal transformation, we obtain
(we write the UV cut off or lattice spacing as ε)

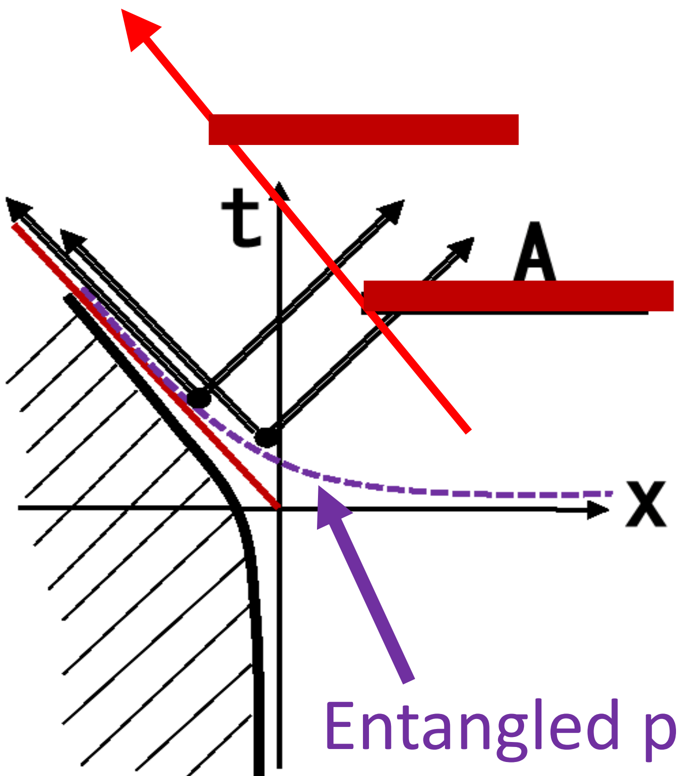
$$S_A = -\frac{\partial}{\partial n}\mathrm{Tr}[(\rho_A)^n] = \frac{c}{6}\log\left[\frac{t + x_0 - p(t - x_0)}{\varepsilon\sqrt{p'(t - x_0)}}\right] + S_{bdy}$$

$$\underset{t\to\infty}{\approx} \frac{c}{12\beta}(t - x_0) + \frac{c}{6}\log\frac{t}{\varepsilon} + S_{bdy} \quad .$$

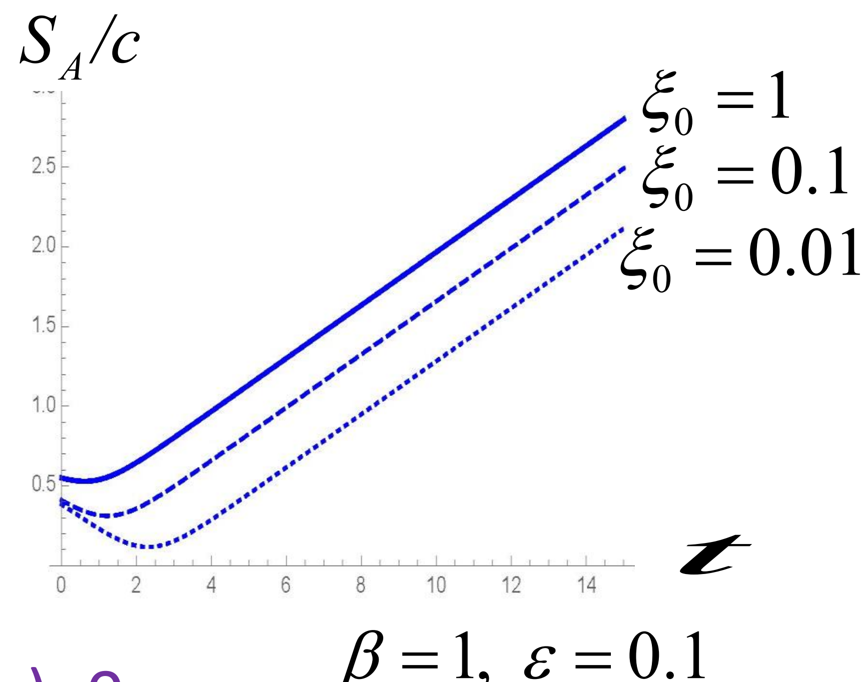
Note that this result is universal for any two dim. CFTs.

It is instructive to choose the time-dependent subsystem:
A=[x0(t),∞], where x0(t)=-t+ξ0 . In this case we find

$$S_A \underset{t\to\infty}{\approx} \frac{c}{6\beta} t + \frac{c}{6}\log\frac{\xi_0}{\varepsilon} + S_{bdy}.$$



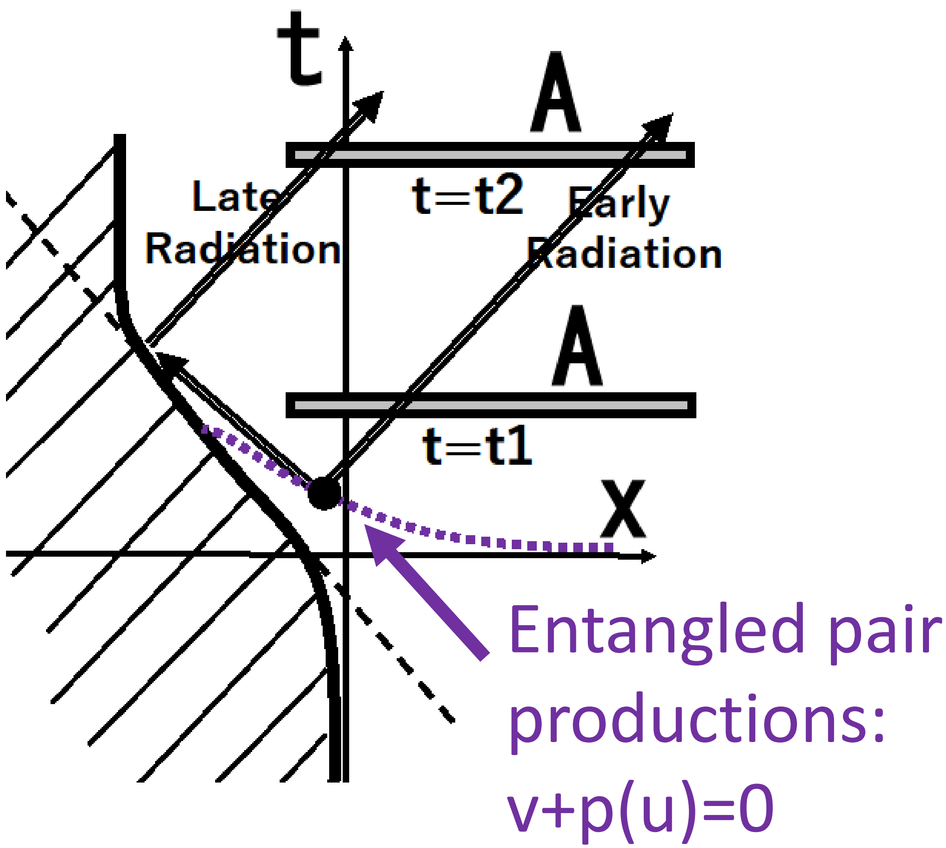
Entangled pair productions: v+p(u)=0

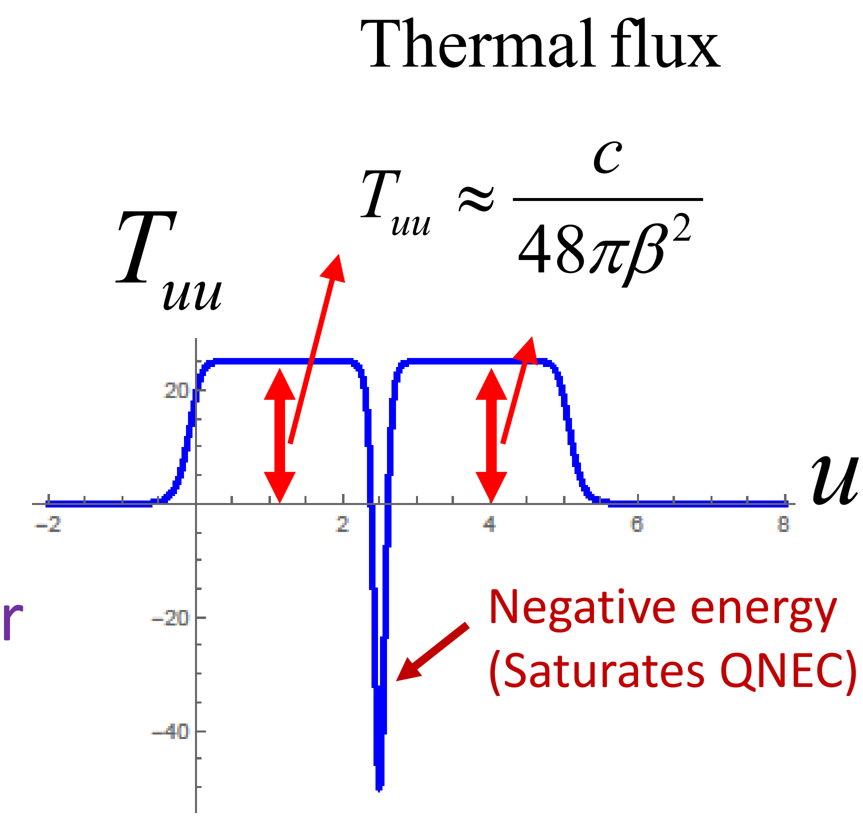


Example 2: Kink Mirror (Model of a BH evaporation)

$$p(u) = -\beta \log(1+e^{-u/\beta}) + \beta \log(1+e^{(u-u_0)/\beta}).$$

$$\Rightarrow Z(t) \underset{t\to-\infty}{\approx} 0, \qquad Z(t) \underset{t\to\infty}{\approx} -u_0/2.$$

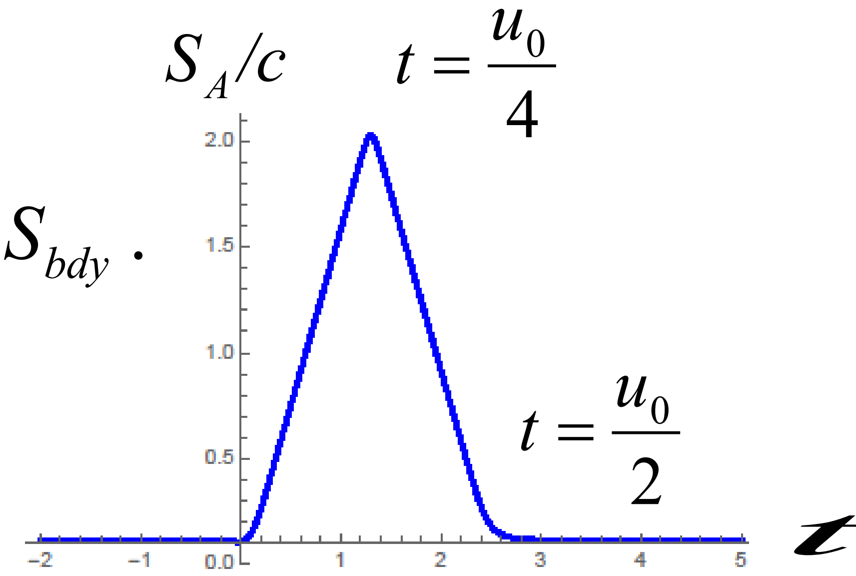




The time evolution of EE

$$S_A = \frac{c}{6}\log\left[\frac{t+x_0-p(t-x_0)}{\varepsilon\sqrt{p'(t-x_0)}}\right]+S_{bdy}\ .$$

reproduces **the perfect page curve** !
(We chose A=[Z(t)+0.1,∞].)

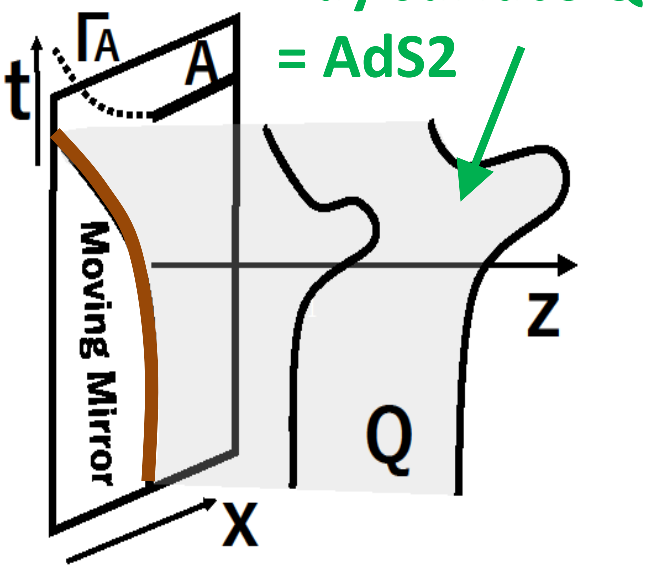


$\beta=1,\ \ \varepsilon=0.1,\ \ u_0=5$

Above results for the semi-infinite subsystem are **universal** in 2d CFTs.
However, the EE for a finite interval A depends on details of 2d CFTs.
➔ Next, we will focus on holographic CFTs.

(3-3) Holographic Moving Mirror

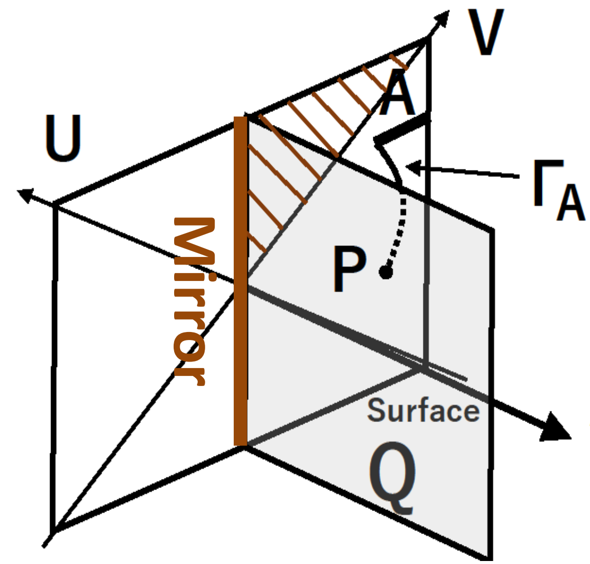
We apply AdS/BCFT to get a gravity dual of moving mirror.



$$\begin{cases} U = p(u) \\ V = v + \dfrac{p''(u)}{2p'(u)} z^2 \\ \eta = z\sqrt{p'(u)} \end{cases}$$

Coordinate transformation

[Banados 1999, Roberts 2012]



$$ds^2 = \frac{dz^2}{z^2} - \frac{dudv}{z^2} + \frac{12\pi}{c} T_{uu}(u) du^2$$

$$ds^2 = \frac{d\eta^2 - dUdV}{\eta^2}$$

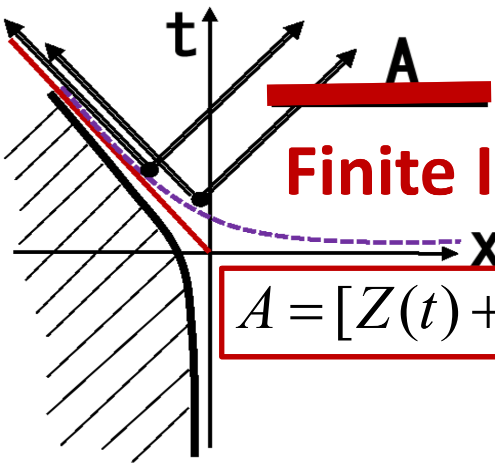
Standard AdS/BCFT setup for BCFT on a half plane

Example 1: Escaping Mirror

$$p(u) = -\beta \log(1 + e^{-u/\beta})$$

The HEE can be computed as

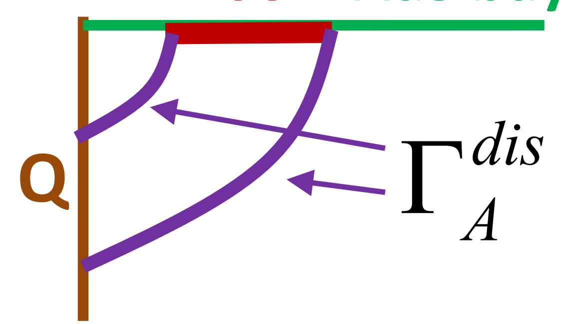
$$S_A = Min\left\{ \frac{L(\Gamma_A^{con})}{4G_N}, \frac{L(\Gamma_A^{dis})}{4G_N} \right\}.$$



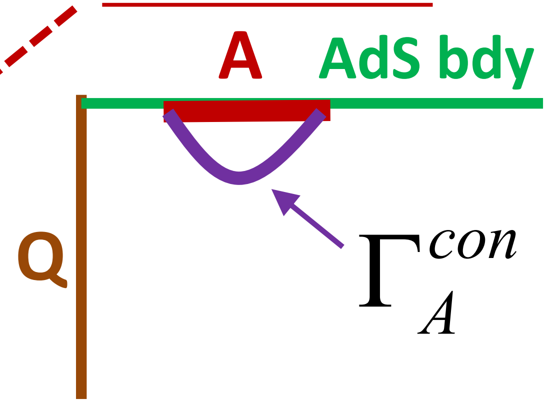
Finite Interval

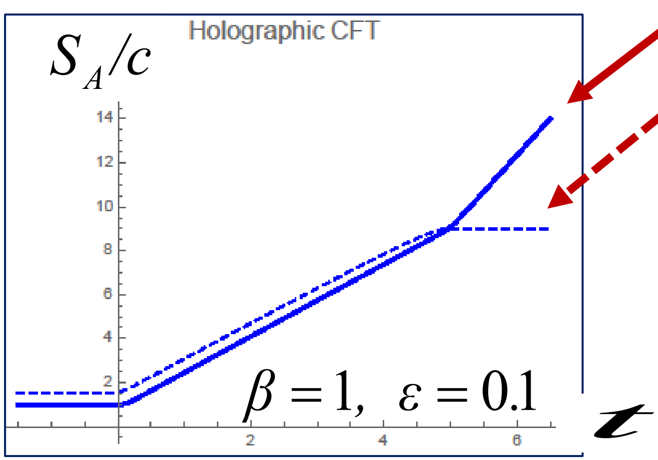
$$A = [Z(t) + 0.1, Z(t) + 10]$$

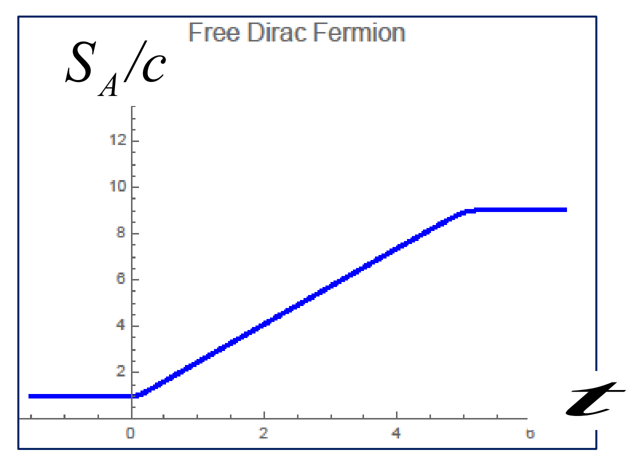
Disconnected



Connected

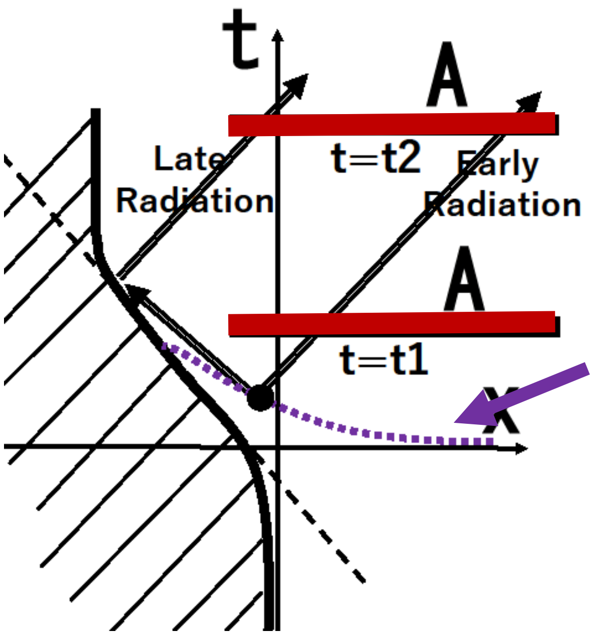






Example 2: Kink Mirror

$$p(u) = -\beta \log(1+e^{-u/\beta}) + \beta \log(1+e^{(u-u_0)/\beta})$$

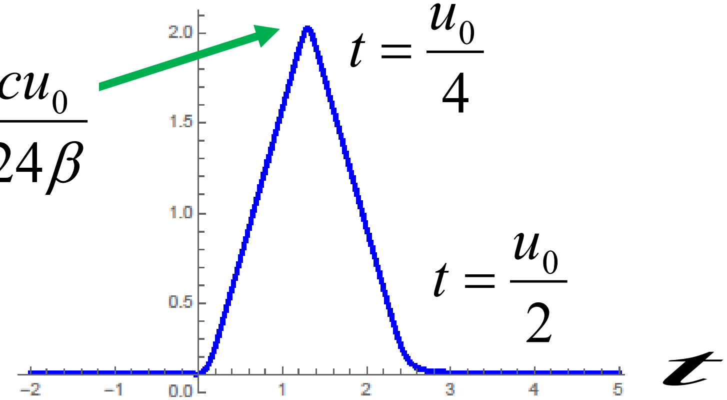


Entangled pair productions: v+p(u)=0

A=Semi infinite interval

$$A = [Z(t)+0.1, \infty]$$

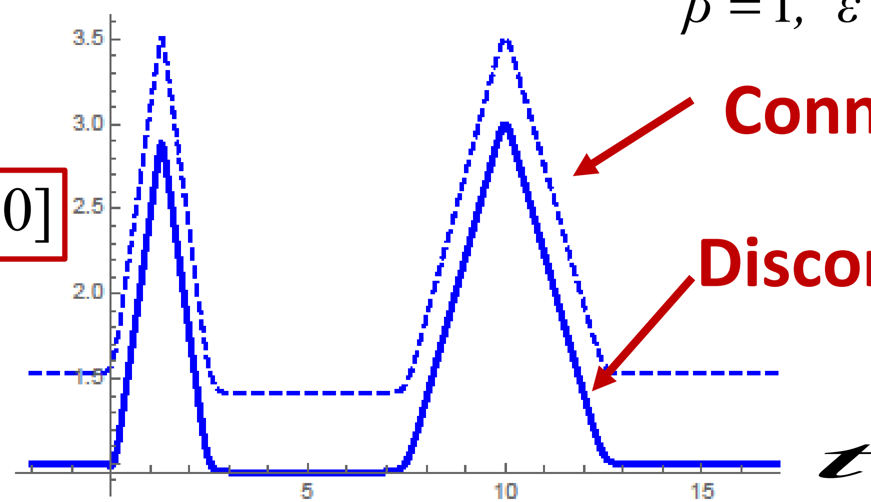
$$S_A = \frac{cu_0}{24\beta}$$



$$\beta = 1, \ \ \varepsilon = 0.1, \ \ u_0 = 5$$

A=Finite Interval

$$A = [Z(t)+0.1, Z(t)+10]$$



Connected Γ_A^{con}

Disconnected Γ_A^{dis}

Thank you very much !