# Algebras, symmetries, entropies

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## Plan of the lectures:

1- Algebraic description of symmetries and generalized symmetries

2- Obstructions to Noether's Theorem, the ABJ anomaly

3- Entropic order parameters. Completeness and modular invariance

# Algebras, Regions, Symmetries

1- QFT is a quantum theory: Hilbert space, operators. Space as a preferred set of subalgebras attached to regions

$$R \longrightarrow \mathcal{A}(R)$$

Algebras of operators are sets of operators with identity, contain all adjoints, the linear combinations and the products close inside the algebra.

Bounded operators in order to have full domain and arbitrary products (bounded spectrum)

Field operators are not operators $\quad \phi(x)|0\rangle \rightarrow \langle 0|\phi(x)\phi(x)|0\rangle = \infty$

Smearing give operators $\quad \phi_\alpha = \int dx^d \, \alpha(x) \, \phi(x)$

We can take $e^{i\,\phi_\alpha}$ or spectral projections to have bounded operators

This is Haag-Kastler description (1964). The information of the theory is in the net of algebras $\mathcal{A}(R)$
Each individual algebra has no information since they are all equal for all QFT and dimension (type III-1 v.n algebras)

“Coordinate free” description, $\phi(x)\,,: \phi(x)^3 :$ give the same theory for a free field.

Conceptually similar to boostrap approach: all fields of a CFT in the description.

Necessity for quantum information measures. Reduced state to a subalgebra.

## 2- von Neumann algebras

$\mathcal{A} \longrightarrow \mathcal{A}'$ commutant $\mathcal{A}' = \{b;\ [b,a] = 0, \forall a \in \mathcal{A}\}$ is an algebra

$\downarrow$

any set of operators

$\mathcal{A}$ is a (v.n.) algebra $\longleftrightarrow$ $\boxed{\mathcal{A} = \mathcal{A}''}$ (von Neumann theorem)

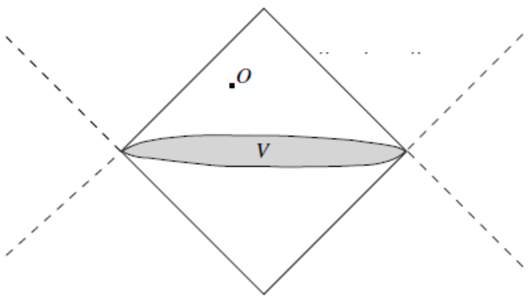
These are all algebras in finite dimensional Hilbert spaces.

In infinite dimensions v.n. algebras are the ones closed under weak topology: physically sound,
The limit of a sequence operators cannot be distinguished from operators in the sequence
by any arbitrary finite set of measurements with finite precision.

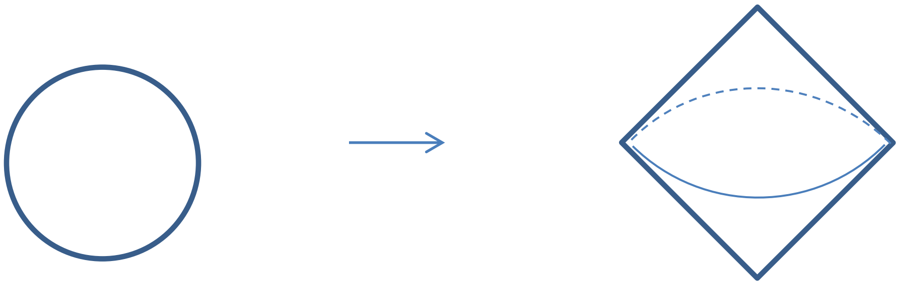
Nice practical purely algebraic way of computing algebras

# 3- Are all algebras for different regions different?

Causal evolution
The existence of a stress tensor



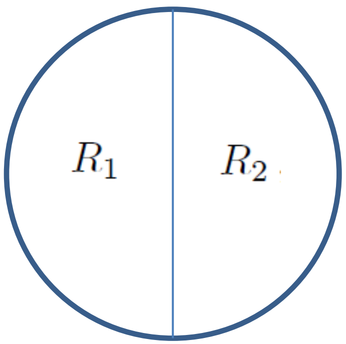

We will assume causal evolution
Only interested in d-1 dimensional pictures (causal regions based at Surface t=0)



## 4- QFT is local: what does it mean $\mathcal{A}(R)$ is assigned to $R$ ?

There are two different meanings to "local"

### 4-a- Additivity: $\mathcal{A}(R)$ is generated by microscopic degrees of freedom in R




$$\mathcal{A}(R_1 \vee R_2) = \mathcal{A}(R_1) \vee \mathcal{A}(R_2)$$

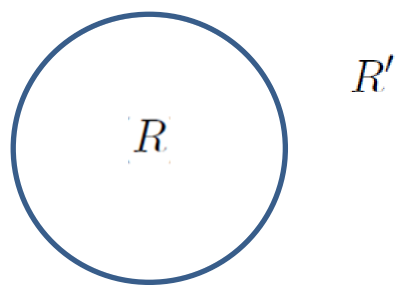
$$\mathcal{A}_1 \vee \mathcal{A}_2 = (\mathcal{A}_1 \cup \mathcal{A}_2)''$$

$$\mathcal{A}(R) \equiv \bigvee_{B \text{ ball}, \cup B = R} \mathcal{A}(B)$$

### 4-b- Causality: spacelike separated observables commute

$$\mathcal{A}(R) \subseteq (\mathcal{A}(R'))'$$

$R'$ causal complement of $R$




We will consider $\mathcal{A}(R)$ both causal and additive

## 5- Completeness = Haag duality of additive net

If $\mathcal{A}(R) \subseteq (\mathcal{A}(R'))'$ but they are not equal, there are operators that could be added to the algebra without ruining causality. This shows a certain incompleteness of the net.

A natural and precise definition of completeness is that the algebra generated by local degrees of freedom (additive algebra) is the maximal one compatible with causality (for any R).

This is Haag duality $\mathcal{A}(R) = (\mathcal{A}(R'))'$ (for the additive net)

This notion of completeness coincides with other notions in the literature (e.g. maximal lattice of charges in gauge theories).

It is equivalent to modular invariance for 2d CFT's

## 6- HDV Haag duality violations

$\mathcal{A}(R) \subsetneq (\mathcal{A}(R'))' = \hat{\mathcal{A}}(R)$ = maximal algebra for R

When this happens $\hat{\mathcal{A}}(R) = \mathcal{A}(R) \vee \{a\}$

Operators a are "non local operators of R": they commute with the local operators outside but cannot be generated by local operators inside R

This implies the same happens dually for the complementary region

## Complementarity diagram

$$
\begin{array}{ccc}
\mathcal{A}(R)\vee\{a\}=\hat{\mathcal{A}}(R) & \supseteq & \mathcal{A}(R) \\
\updownarrow' & & \updownarrow' \\
\mathcal{A}(R') & \subseteq & \hat{\mathcal{A}}(R')=\mathcal{A}(R)\vee\{b\}
\end{array}
$$

## Non local operators come in dual pairs

If in one line the algebras coincide then the same happens in the other by v.n theorem.

The a's and b's cannot commute all of them to each other. Otherwise $\hat{\mathcal{A}}(R)\subseteq(\hat{\mathcal{A}}(R'))'=\mathcal{A}(R)$

Therefore, the maximal algebras do not form a good causal net.

If a is non local in R the same happens for combinations with additive operators $\sum_i \mathcal{O}_i^1\, a\, \mathcal{O}_i^2$

This divides the set of non local operators in classes under the action of local ones.

## Examples

1- Free Maxwell field in d=3 $\longleftrightarrow$ derivatives of a massless scalar $F_{\mu\nu} = \epsilon_{\mu\nu\sigma}\partial^\sigma \phi$ $\qquad \mathcal{A}_F = \mathcal{A}_{\partial\phi}$

$$\phi(x) \to \psi_q = e^{i\,q\,\int \alpha(x)\,\phi(x)}\,, \qquad \int \alpha = 1$$

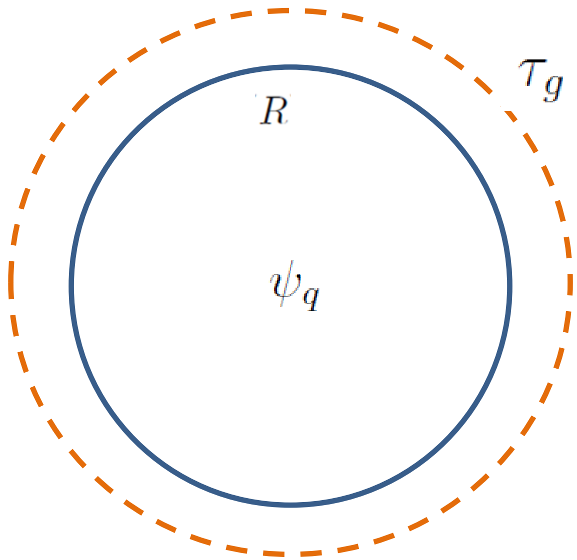
$$\partial^\mu(\partial_\mu \phi) = 0 \to j^\mu = \partial^\mu \phi$$ conserved current $$Q = \int \beta(x)\, j^0(x) = \int \beta(x)\,\dot{\phi}\,, \quad \beta(x) = 1\,, x \in R$$

$$\tau_g = e^{i\,g\,Q}$$ twist operator of the symmetry

$$[Q, \partial\phi] = 0 \qquad [Q, \phi(x)] = i \to \tau_g\, \phi(x)\, \tau_g^\dagger = \phi(x) + g$$

$$\tau_g\, \psi_q\, \tau_g^\dagger = e^{i\,q\,g}\psi_q$$

$$\psi_{q1}\,\psi_{q2} = \psi_{q1+q2} \qquad \tau_{g1}\,\tau_{g2} = \tau_{g1+g2}$$




It may seem we could generalize to a theory F with a global symmetry G, and take O=F/G, the neutral operators, as additive algebra. But if the symmetry is unbroken, no charged operators exist in the neutral Hilbert space and the twist is local outside the ball: $\tau_g(R)\,\tau_g(R') = g \equiv 1$
HDV for balls occur for spontaneously broken global symmetries $\langle \psi_q \rangle \neq 0$
The present case is a Goldstone boson .

$$\hat{\mathcal{A}}(R) = \mathcal{A}(R) \vee \{\psi_q\,, q \in \mathbb{R}\}$$
$$\hat{\mathcal{A}}(R') = \mathcal{A}(R') \vee \{\tau_g\,, g \in \mathbb{R}\}$$

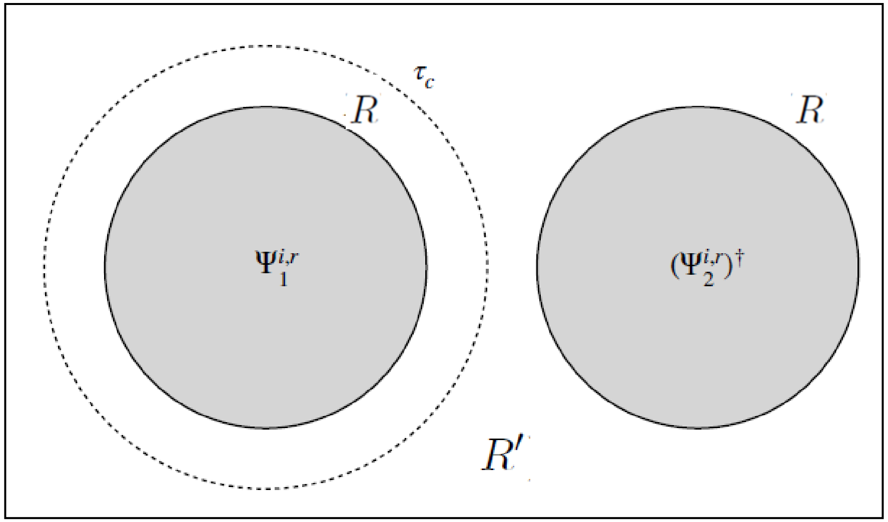
2- Unbroken global internal symmetries, regions with non trivial homotopy groups $\pi_0$ or $\pi_{d-2}$

Neutral part of a global symmetry $\mathcal{O} = \mathcal{F}/G$ Take only operators invariant under the group G

Non local operators:

Charge-anticharge operators (representations) $\mathcal{I}_r = \sum_i \psi_1^{i,r} (\psi_2^{i,r})^\dagger$

Twists (conjugacy classes) $\tau_c = \sum_{h\in c} \tau_h$

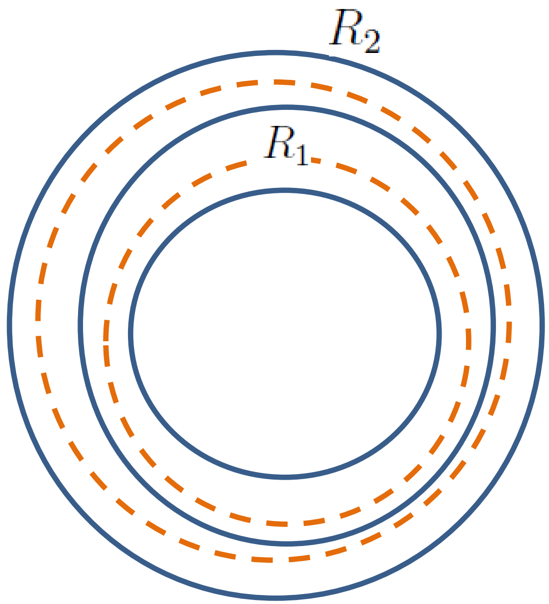



$$(\mathcal{O}(R))' = \mathcal{O}(R') \vee \{\tau_c\}$$
$$(\mathcal{O}(R'))' = \mathcal{O}(R) \vee \{\mathcal{I}_r\}$$

SSB also has HDV for disconnected regions
and their complements

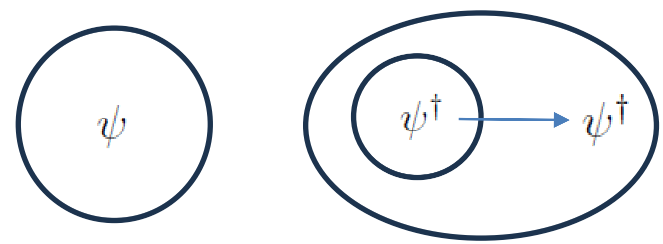
## Commentaries

### a) Topology




"transportability", topological operators

$$\hat{\mathcal{A}}(R_2) = \hat{\mathcal{A}}(R_1) \vee \mathcal{A}(R_2 - R_1)$$
$$\hat{\mathcal{A}}(R_1') = \hat{\mathcal{A}}(R_2') \vee \mathcal{A}(R_2 - R_1)$$




HDV are contagious: propagate from a single region to any deformed, topologically equivalent one,  in an isomorphic way.

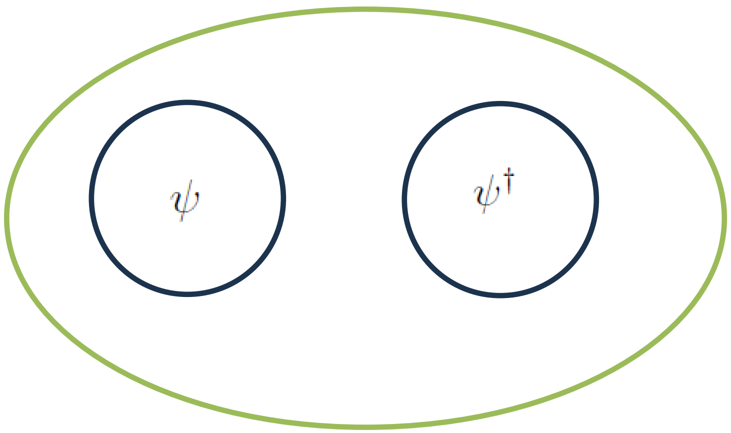
Determines a property of the whole net, not just a region.

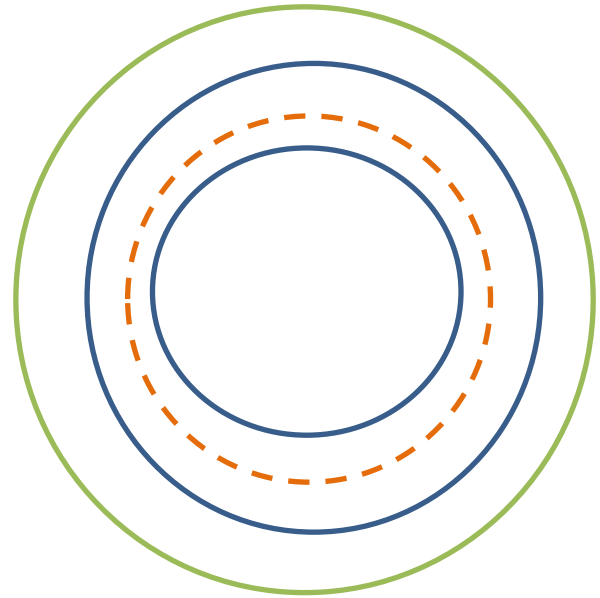
Relation between topologies of R with possible types of HDV.

Contaminates all scales and RG flow (as happens with ordinary symmetries as a particular case).

Use for partially classify theories, order parameters.

b) "Non-locality" of a non local operator



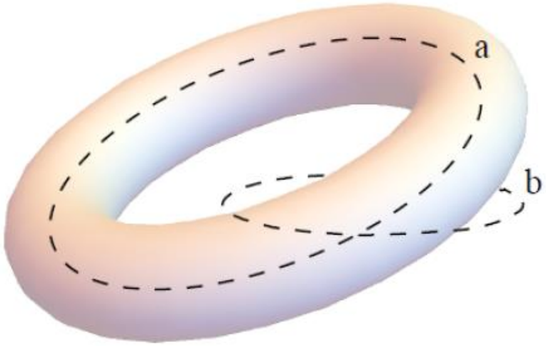



"Non local operator" is a notion relative to a region. A non local operator for a topologically non trivial region R is generally additive for a ball containing R.

Under Haag duality for balls all operators are ultimately additive for a large enough ball.

## Examples

### 3- Maxwell field in d>3




Wilson loops $W = e^{i\,q\,\oint A_\mu\, dx^\mu} = e^{i\,q\,\int d\sigma\, \bar{B}\cdot\bar{n}}$

't Hooft loops $T = e^{i\,q\,\int d\sigma\, \bar{E}\cdot\bar{n}}$

These operators commute with local operators outside the torus because of conservation of the fluxes $\nabla E = \nabla B = 0$
but cannot be written in terms of the electric and magnetic fields (gauge invariant operators) in the torus
They are non local in the torus

However, they can be written as a flux of the magnetic field on a surface and then is an additive operator on a ball that includes the torus

$$W_q = e^{i\,q\,\Phi_B} \qquad T_q = e^{i\,q\,\Phi_E} \qquad [\Phi_B, \Phi_E] = i$$

$$W_q W_{q'} = W_{q+q'}\,, \quad T_g T_{g'} = T_{g+g'} \qquad W_q T_g = e^{i\,q\,q}\, T_g W_q$$

gives $A = \mathbb{R}\,, B = \mathbb{R}\,, d \neq 4\,,\qquad A = \mathbb{R}^2\,, B = \mathbb{R}^2\,, d = 4$

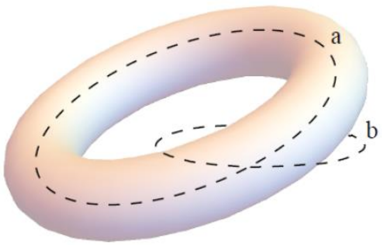
In general gauge theories can give HDV for regions with non trivial $\pi_1$ or $\pi_{d-3}$

In d=3 this coincides with HDV produced by global symmetry ( $\pi_0$ or $\pi_{d-2}$ )

Commentaries

a) One can add non-local operators to the additive algebra but still keeping it causal, and eventually reach Haag duality. This type of maximal algebras can be called Haag-Dirac nets

$$\mathcal{A}(R) \subset \mathcal{A}_{HD}(R) \subset \hat{\mathcal{A}}(R)$$




causal, additive
not Haag dual

causal, Haag dual
not additive

not causal

Haag duality reached at the expense of additivity, not the two together: not complete

No physical difference in these choices: operators are already in the additive algebra of balls

Could be several possible choices

The physics of HDV is equivalent to existence of different causal nets for the same theory

b) For the Maxwell field the HD nets are given by solutions of the Dirac quantization condition (maximal lattice of commuting charges)

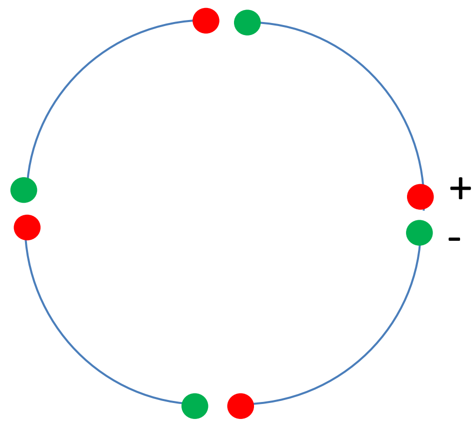
$$(q,g) = n(q_1,g_1) + m(q_2,g_2)$$

$$q_1\, g_2 - q_2\, g_1 = 2\pi$$

$$W_q T_g = e^{i\,q\,q}\, T_g W_q$$

# Examples

## 4- QED




$$\psi(x)\, e^{i\, q_0\, \int_x^y dx^\mu\, A_\mu(x)}\psi^\dagger(y)$$

Wilson lines

Wilson loop is broken into additive operators along the loop

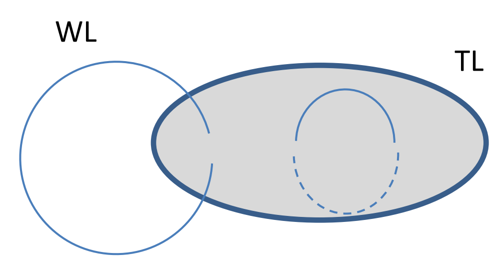
Charges of classes of non local WL are now in $q \in [0, q_0)$ . This is R $\rightarrow$ R/Z=U(1). This is still continuous and the generator is still the magnetic flux that is still conserved.

Electric flux is not conserved anymore. TL that do not commute with the broken WL cannot be non local operator. Then the TL charges are now integers $g = \dfrac{2\pi k}{q_0}$

$$A = U(1)\,, B = Z\,, d \neq 4\,, \qquad A = U(1)\times\mathbb{Z}\,, B = \mathbb{Z}\times U(1)\,, d = 4$$

Construction of the TL as a singular gauge transformation

´t Hooft (1978) $\quad \begin{array}{l} A_\mu \to A_\mu + \partial_\mu \alpha \\ \oint dx^\mu\, \partial_\mu \alpha = g \end{array} \quad \rightarrow \quad W_q\, T_g = e^{i\, q\, q}\, T_g\, W_q$



Commentaries

a) If we add electric and magnetic charges the respective loops become additive, and then they must be commuting for linked loops. Charges have to obey Dirac quantization condition.

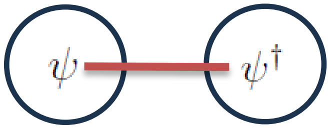
b) Choosing a maximal lattice of electric and magnetic charges no HDV remains. The theory becomes complete. Equivalence of completeness as Haag duality of the additive net with usual completeness.

c) The electromagnetic current is conserved but does not define a global symmetry in the previous local sense:

$$\partial_\mu j^\mu = 0\,, \quad j^\mu = \partial_\nu F^{\mu\nu} \longrightarrow$$ it is a total derivative

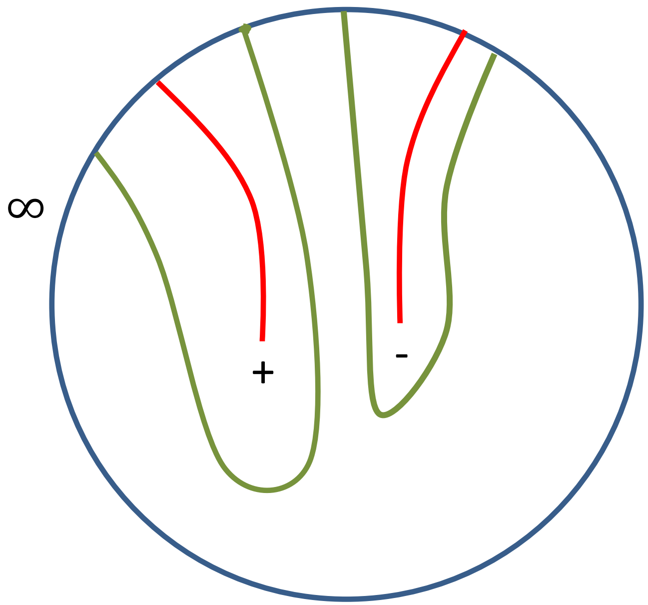
$$Q = \int_V j_0 = \int_{\partial V} E.n \longrightarrow$$ charge and twist are additive operators on the boundary

In the same way charge-anticharge operators are not a two-ball operator because of Wilson line




d) For a continuous set of non local operators one can think (in analogy with Noether's theorem) that there should be a charge generator, a conserved flux given by the integral of a closed form-field. However, a closed form-field does not produce a HDV if it is exact: e.g. the above current for gauge charges, $dF = 0$ but $F \neq dA$ for a physical field A. This is the case of the electromagnetic field but not the case of the massive vector field.
Closed+exact is trivial. HDV a "quantum" version of cohomology that is not restricted to continuum form fields.

e) However, gauge charges can give place to another type of HDV, for infinite cone-like regions:




Buchholtz, Fredenhagen (1982)

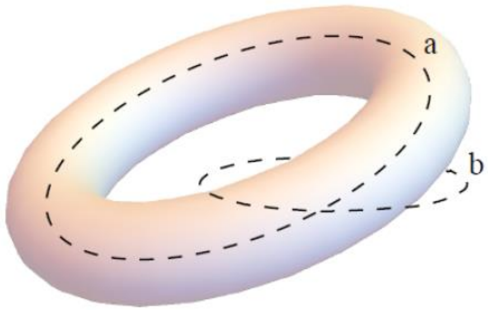
The dual operators are electric or magnetic fluxes at infinity

Thought in the boundary gives the same type of HDV sectors for disconnected regions as global symmetries.
(Some) topological theories that can be put as the IR limit of a theory with local degrees of freedom are of this type.

## Example

### 5- Non-Abelian gauge field

Regions with non trivial homotopy groups $\pi_1$ or $\pi_{d-3}$




Wilson loops $W_r \equiv \mathrm{Tr}_r \mathcal{P}\, e^{i \oint_C dx^\mu A_\mu^r}$

Not all Wilson loops are non locally generated in a ring-like region: for charged representations the loops can be broken by Wilson lines ended in charged operators

$$\phi_r(x)\, P e^{i\int_x^y dx^\sigma A_\sigma}\, \phi_r^\dagger(y) \qquad F_{\mu\nu}(x)\, P e^{i\int_x^y dx^\sigma A_\sigma}\, F_{\alpha\beta(y)} \longrightarrow$$

Representations generated by the adjoint are locally generated

For pure gauge groups only non trivial representations of the center of the group are non local.

Dual operators are the 't Hooft loops labelled by the center of the group

$$T^z W^r = \chi_r(z)\, W^r T^z$$

Non local operators correspond to the center of the group and its representations (the dual group). Commutation relations are fixed.

Character of the representation

For SU(N) the dual groups of non local operators are $Z_N$

## Commentaries

### a) SU(2) versus SO(3) pure gauge theory

In a lattice these two groups give different models, with and without HDV (a $Z_2$ group for SU(2)).

What happens in the continuum for these two models depends on the continuum limit: they are both described by the same Lie algebra that determines the Lagrangian. It could be (or not) that there is another field in the fundamental representation that eliminates the HDV.

However, in the generalized symmetry literature the names SU(2) or SO(3) pure gauge theory are used in a different way: an SU(2) theory is one that contains the WL and an SO(3) theory one containing the TL. One is though as a "genuine line operator" and the other as a "topological surface operator" (say one is the charge and the other the symmetry operator).

As we have seen both dual non local operators exist at the same time or none of them, they have to be taken on equal footing. Then this naming corresponds to different HD nets. Different HD nets correspond to the same physical theory.

Both operators are line operators in the sense they commute with additive operators outside, both are surface operators in the sense that to write them in terms of additive operators we need surfaces.

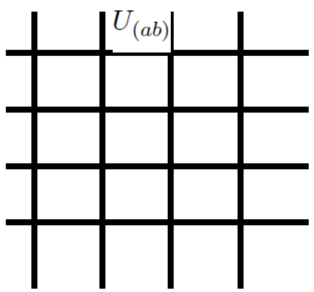
The same commentary applies to the "compact free Maxwell field" that is the usual free Maxwell field (with non compact RxR sectors) with a particular choice of HD net.

b) Haag duality for balls implies the TL and WL are additive in balls. How can we see this more explicitly?

H. C, M. Huerta, J. Magan, D. Pontello (2020)

In the lattice it can be checked
(there are "non-Abelian Stokes theorem" expressions but are not local gauge invariant expressions):

$$U'_{(ab)} = g_a U_{(ab)} g_b^{-1} =: U^g_{(ab)}$$ gauge transformation




$$\Psi[U] = \Psi[U^g]$$ gauge invariant Hilbert space

$$(W^r_\Gamma \Psi)[U] \ = \chi_r(U_{(a_1 a_2)} U_{(a_2 a_3)} ... U_{(a_k a_1)}) \Psi[U]$$ WL (magnetic) operators

$$(L^g_l \Psi)[U_1, ..., U_{L_N}] = \Psi[U_1, ..., gU_l, ..., U_{L_N}]$$

$$E^c_l := \sum_{h\in c} L^h_l$$ electric link operators

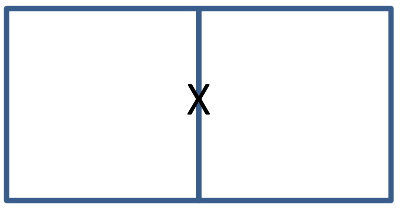
$$T^z_\Sigma := \prod_{l \perp \Sigma} E^z_l$$ TL is directly seen as additive in surfaces

$$E^r_l := \frac{d_r}{|G|} \sum_{g\in G} \chi^*_r(g)\, L^g_l\,, \quad (E^r_l)^\dagger = E^r_l\,, \quad E^r_{\bar{l}} = E^{\bar{r}}_l\,, \quad E^r_l E^{r'}_l = \delta_{rr'} E^r_l\,, \quad \sum_r E^r_l = \mathbf{1}$$

electric projectors on the different representations

$$W^r_{(\Gamma_1 \Gamma_2)} = d_r \sum_{r'} E^{r'}_l\, W^r_{(\Gamma_1 l)} W^r_{(\bar{l} \Gamma_2)} E^{r'}_l$$

sewing two WL
with electric operators



## General results about the possible structure of HDV

1) For d>2 HDV for a ball or disconnected regions and duals (non trivial $\pi_0$ or $\pi_{d-2}$ ) come from the examples of neutral algebras under global internal symmetries, broken or unbroken. The group can be abelian or non-Abelian.

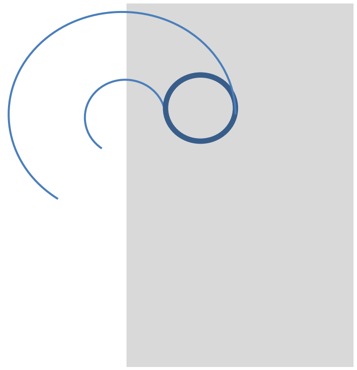
The theory can be extended (to a new additive algebra, do not change correlators nor the Hamiltonian) in a unique way such that the extended theory does not have this type of HDV: O $\longrightarrow$ F, O=F/G. The extended theory can be fermionic.

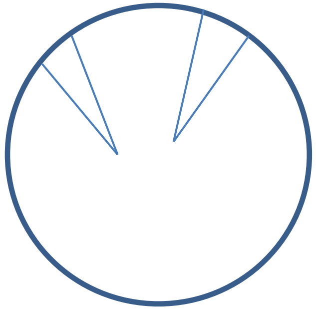
This follows from DHR theorem (Doplicher, Haag, Roberts 1971-1989)

2) For d=2, HDV for multi-intervals, are related to categories that can be non invertible with braid group statistics. Maximal causal extensions (modular invariant CFT's) of the additive algebra can be usually achieved (rational CFT's) but the extension are generally non-unique.

3) For d>3, once we have completed two ball HDV, the HDV for rings of any dimension and their duals ($\pi_k$ , $1 \leq k \leq d-3$) are given by dual Abelian groups (commutation relations are fixed). This follows from a dimensional reduced version of DHR theorem (H. C, M. Huerta, J. Magan, D. Pontello (2020), HC, J. Magan, in preparation). The additive algebra cannot be extended to eliminate these type of HDV.

4) Topological cone-like cases behave as in a one dimension less. Disjoint cone sectors can be related to non-invertible symmetries in d=3 and are group-like for d>3.

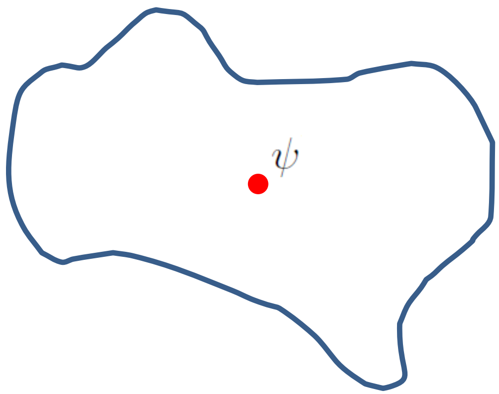




## Standard approach to generalized symmetries

Gaiotto, Kapustin, Seiberg, Willett (2015)

Topological maps: can be freely moved until a charge is crossed




$$\psi' = \rho(\psi)$$

In this case is a 0-form symmetry, maps represented by codimension 1 surface (in Euclidean spacetime, twist and antitwist, no boundary)

Different types of charges of dimension k associated to codimension k+1 topological maps, k-form symmetry

"Simple maps" have labels, and can be composed to give fusion categories

$$\rho_i \circ \rho_j = \oplus_k N_{ij}^k \, \rho_k$$

Invertible if $\rho \circ \bar{\rho} = 1$ , if there are more terms is non-invertible

Usually assumed the stress tensor is unaffected by the maps: there is a subalgebra of unaffected operators

## Connection with different presentations of symmetries: Algebra inclusions and endomorphisms

An inclusion of algebras is associated to a category of endomorphisms:

$$\rho(\mathcal{A}) \subset \mathcal{A} \qquad \rho(ab) = \rho(a)\rho(b) \qquad \rho(\mathbb{1}) = \mathbb{1} \qquad \rho(a^\dagger) = \rho(a)^\dagger$$

$$\mathcal{A} \subset \mathcal{B}$$

Endomorphisms related by unitaries in $\mathcal{A}$ are taken as equivalent (sectors).

See Longo,Rehren (1994)

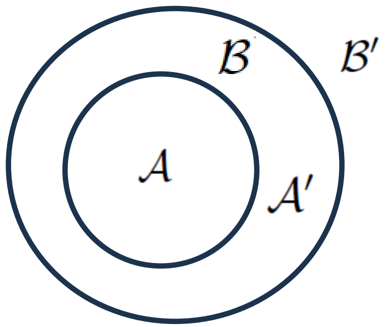
$$\rho \sim \rho_U \qquad \rho_U = U\rho U^\dagger$$

Given the inclusion, there are non trivial endomorphisms associated to "charged" operators in $\mathcal{B}$

$$\psi a = \rho(a)\psi \ , \quad a \in \mathcal{A} \quad \psi \in \mathcal{B}$$

E.g. the canonical endomorphism produced by Tomita-Takesaki reflections:

$$\rho(\mathcal{A}) \equiv \ j_{\mathcal{A}} j_{\mathcal{B}}(\mathcal{A}) \subset \mathcal{A}$$




Endomorphisms can be composed to give other endomorphisms.

Natural way to define direct sums and decompositions in direct sums of endomorphisms. Definition of irreducible endomorphisms

$$\mathbb{1} = \sum_r \omega_r \omega_r^\dagger \,, \quad \omega_r^\dagger \omega_{r'} = \delta_{rr'} \mathbb{1} \,, \quad \omega_r \in \mathcal{A}$$

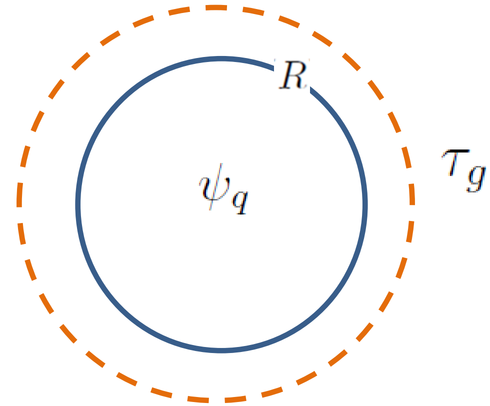
$$\rho \simeq \oplus_r \rho_r \qquad \Rightarrow \qquad \rho(a) \equiv \sum_r \omega_r \, \rho_r(a) \, \omega_r^\dagger, \qquad \omega_r, a \in \mathcal{A}$$

Fusion rules of irreducible sectors

$$\rho_i \circ \rho_j = \oplus_k N_{ij}^k \, \rho_k$$

There is always the dual inclusion and dual endomorphisms $\quad \mathcal{B}' \subseteq \mathcal{A}'$

Three versions of symmetry




## HDV

$$\hat{\mathcal{A}}(R) = \mathcal{A}(R) \vee \{\psi_q \,, q \in \mathbb{R}\}$$
$$\hat{\mathcal{A}}(R') = \mathcal{A}(R') \vee \{\tau_g \,, g \in \mathbb{R}\}$$

Generalized symmetry

$$\begin{array}{ccc} \hat{\mathcal{A}}(R) & \supset & \mathcal{A}(R) \\ \updownarrow' & & \updownarrow' \\ \mathcal{A}(R') & \subset & \hat{\mathcal{A}}(R') \end{array}$$

Dual generalized symmetry

"0-form symmetries"
Topological endomorphisms

$$\tau\, x = \rho(x)\, \tau$$

DHR endomorphisms
"localizable, transportable".
generated by charges $\psi a = \rho(a) \psi$
Fusion given by representations of groups

Generalized symmetries come in dual pairs. Subalgebras appear from invariant operators under endomorphism.

0-form symmetries for d>2 are invertible (come from groups). $\rho(x) = \tau\, x\, \tau^\dagger$

Combinations of endomorphisms can be non invertible $E(x) = |G|^{-1} \sum_{g\in G} \tau_g\, x\, \tau_g^\dagger$

$$\mathcal{O}_1 \subset \mathcal{O}_2 \subset F$$
H
G

DHR theorem (perhaps better remembered as a "spin-statistics theorem") is also a classification of 0-form symmetries.

Statistics of sectors: permutation group → integer dimensions → group representations