General results about the possible structure of HDV

1) For d>2 HDV for a ball or disconnected regions and duals come from the examples of neutral algebras under global internal symmetries, broken or unbroken. The group can be abelian or non-Abelian.

The theory can be extended (to a new additive algebra, do not change correlators nor the Hamiltonian) in a unique way such that the extended theory does not have this type of HDV: O=F/G. The extended theory can be fermionic.

This follows from DHR theorem (Doplicher, Haag, Roberts 1971-1989)

2) For d=2, HDV for multi-intervals, are related to categories that can be non invertible with braid group statistics. Maximal causal extensions (modular invariant CFT's) of the additive algebra can be usually achieved (rational CFT's) but the extension are generally non-unique.

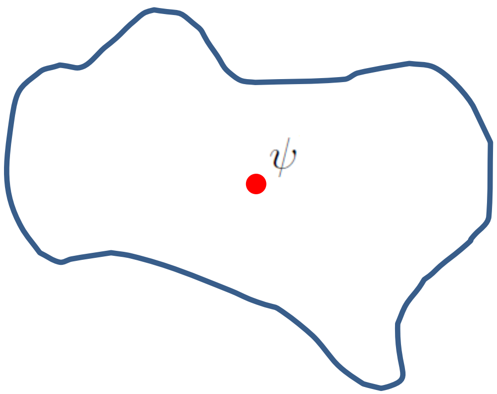
3) For d>3, once we have completed two ball HDV, the HDV for rings of any dimension and their duals ($\pi_k \, , \quad 1 \leq k \leq d-3$) are given by dual Abelian groups (commutation relations are fixed). This follows from a dimensional reduced version of DHR theorem (H. C, M. Huerta, J. Magan, D. Pontello (2020), HC, J. Magan, in preparation). The additive algebra cannot be extended to eliminate these type of HDV.

4) Topological cone-like cases behave as in a one dimension less. Disjoint cone sectors can be related to non-invertible symmetries in d=3 and are group-like for d>3.

Standard approach to generalized symmetries

Gaiotto, Kapustin, Seiberg, Willett (2015)

Topological maps: can be freely moved until a charge is crossed



$$\psi' = \rho(\psi)$$

In this case is a 0-form symmetry, maps represented by codimension 1 surface (in Euclidean spacetime, twist and antitwist, no boundary)

Different types of charges of dimension k associated to codimension k+1 topological maps, k-form symmetry

"Simple maps" have labels, and can be composed to give fusion categories

$$\rho_i \circ \rho_j = \oplus_k N_{ij}^k \, \rho_k$$

Invertible if $\rho \circ \bar{\rho} = 1$, if there are more terms is non-invertible

Usually assumed the stress tensor is unaffected by the maps: there is a subalgebra of unaffected operators

Connection with different presentations of symmetries: Algebra inclusions and endomorphisms

An inclusion of algebras is associated to a category of endomorphisms:

$$\rho(\mathcal{A}) \subset \mathcal{A} \qquad \rho(ab) = \rho(a)\rho(b) \qquad \rho(\mathbb{1}) = \mathbb{1} \qquad \rho(a^\dagger) = \rho(a)^\dagger \qquad\qquad \mathcal{A} \subset \mathcal{B}$$

Endomorphisms related by unitaries in $\mathcal{A}$ are taken as equivalent (sectors).

$$\rho \sim \rho_U \qquad \rho_U = U \rho U^\dagger$$

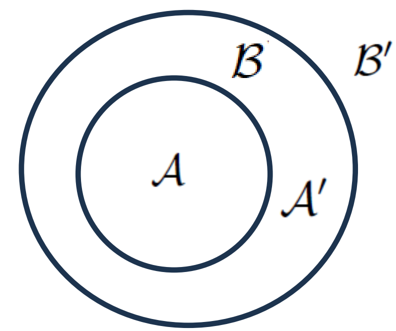
See Longo,Rehren (1994)

Given the inclusion, there are non trivial endomorphisms associated to "charged" operators in $\mathcal{B}$

$$\psi a = \rho(a)\psi \, , \quad a \in \mathcal{A} \quad \psi \in \mathcal{B}$$

E.g. the canonical endomorphism produced by Tomita-Takesaki reflections:

$$\rho(\mathcal{A}) \equiv j_{\mathcal{A}} j_{\mathcal{B}}(\mathcal{A}) \subset \mathcal{A}$$



Endomorphisms can be composed to give other endomorphisms.

Natural way to define direct sums and decompositions in direct sums of endomorphisms. Definition of irreducible endomorphisms

$$\mathbb{1} = \sum_r \omega_r \omega_r^\dagger \, , \quad \omega_r^\dagger \omega_{r'} = \delta_{rr'} \mathbb{1} \, , \quad \omega_r \in \mathcal{A}$$

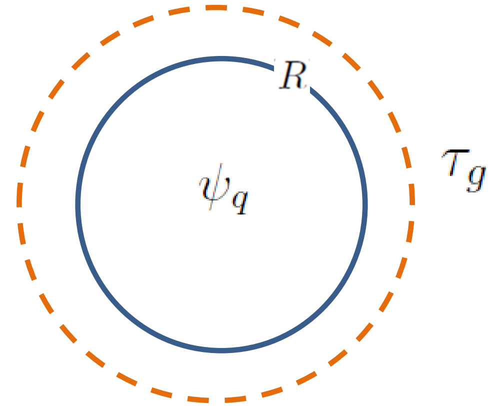
$$\rho \simeq \oplus_r \rho_r \qquad \Rightarrow \qquad \rho(a) \equiv \sum_r \omega_r \, \rho_r(a) \, \omega_r^\dagger , \qquad \omega_r, a \in \mathcal{A}$$

Fusion rules of irreducible sectors

$$\rho_i \circ \rho_j = \oplus_k N_{ij}^k \, \rho_k$$

There is always the dual inclusion and dual endomorphisms $\quad \mathcal{B}' \subseteq \mathcal{A}'$

Three versions of symmetry



HDV

$$\hat{\mathcal{A}}(R) = \mathcal{A}(R) \vee \{\psi_q \, , q \in \mathbb{R}\}$$
$$\hat{\mathcal{A}}(R') = \mathcal{A}(R') \vee \{\tau_g \, , g \in \mathbb{R}\}$$

$$\begin{array}{ccc} \hat{\mathcal{A}}(R) & \supset & \mathcal{A}(R) \\ \updownarrow' & & \updownarrow' \\ \mathcal{A}(R') & \subset & \hat{\mathcal{A}}(R') \end{array}$$

Generalized symmetry

"0-form symmetries"
Topological endomorphisms

$$\tau \, x = \rho(x) \, \tau$$

Dual generalized symmetry

DHR endomorphisms
"localizable, transportable",
generated by charges $\psi a = \rho(a) \psi$
Fusion given by representations of groups

Generalized symmetries come in dual pairs.

0-form symmetries for d>2 are invertible (come from groups). $\rho(x) = \tau \, x \, \tau^\dagger$

Combinations of endomorphisms can be non invertible $E(x) = |G|^{-1} \sum_{g \in G} \tau_g \, x \, \tau_g^\dagger$

H

$$\mathcal{O}_1 \subset \mathcal{O}_2 \subset F$$

G

DHR theorem (perhaps better remembered as a "spin-statistics theorem") is also a classification of 0-form symmetries. (Statistics of sectors: permutation group - integer dimensions -group representations)

ABJ anomaly as a U(1) symmetry, and Noether's theorem

Based on works with Valentin Benedetti and Javier Magán:

https://arxiv.org/abs/2309.03264
https://arxiv.org/abs/2205.03412

Noehter's theorem (weak form): there are local implementations for any global symmetry (twists)
Quite generally true, even for discrete symmetries (Doplicher, Longo (1983,1984), Buchholz, Doplicher, Longo (1986))

Split property (no Hagedorn transition)

$\mathcal{A}_A \subseteq \mathcal{N} \subseteq (\mathcal{A}_B)'$ Intermediate type I factor

$\mathcal{H} = \mathcal{H}_{\mathcal{N}} \otimes \mathcal{H}_{\mathcal{N}'}$

Modular reflection

$\mathcal{A}\,,|0\rangle \to J\,,\, J\,\mathcal{A}\,J = \mathcal{A}'$

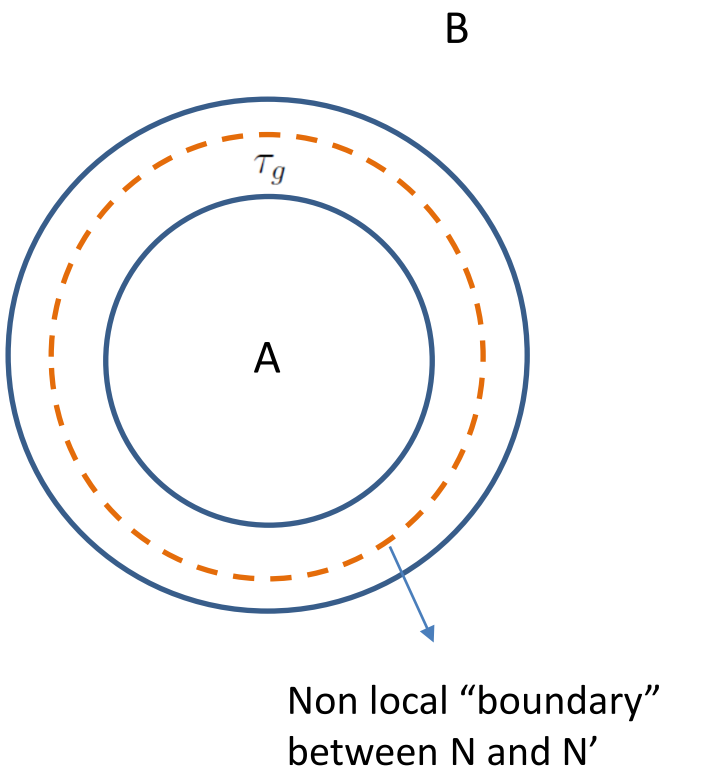
Standard split using modular reflection J

$\mathcal{N}_{AB} \equiv \mathcal{A}_A \vee J_{AB}\mathcal{A}_A J_{AB}\,, \quad \text{or} \quad \mathcal{N}'_{AB} = \mathcal{A}_B \vee J_{AB}\mathcal{A}_B J_{AB}$

Standard twists

$\tau_g \otimes \tau'_g \qquad \tau_g \in \mathcal{N}_{AB},\, \tau'_g \in \mathcal{N}'_{AB}$

$\tau_g \tau_h = \tau_{gh}\,, \quad g\tau_h g^{-1} = \tau_{ghg^{-1}}$



For a continuous symmetry with a Noether current we could also use (in this sense is a weak form of Noether's theorem) $\tau = e^{i\int \alpha(x)\, j_0(x)}$

Noether's theorem (strong form): for any continuous symmetry there is a conserved current.

Some examples of violations of (strong) Noether's theorem:

Free graviton
No stress tensor.

Two free Maxwell fields
No current for rotation symmetry.

Duality symmetry Maxwell field
No duality current.

Free and massless. Why?

Maxwell field for dimension $d \neq 4$
Derivatives of free scalar $d>2$
No dilatation current.

Known counterexamples
for DI implies CI

ABJ anomaly → Would be interacting case.
No chiral current

At the practical level from a Lagrangian one gets non gauge invariant currents that cannot be improved

$$j^\mu = A^1_\nu \, F_2^{\nu\mu} - A^2_\nu \, F_1^{\nu\mu}$$

ABJ anomaly:

Massless fermions coupled to electromagnetic field, chiral symmetry $\psi \to e^{-i\alpha\gamma^5}\psi$

$j_5^\mu = \bar{\psi}\gamma^5\gamma^\mu\psi$ would be Noether current, classically conserved

$\partial_\mu j_5^\mu = \frac{1}{16\pi^2}\epsilon^{\mu\nu\alpha\beta}F_{\mu\nu}F_{\alpha\beta}$ quantum non conservation

Necessary to explain neutral pion decay into photons in QCD

Several derivations in Lagrangian QFT: Feynman diagrams, non invariance of the path integral measure.

Is this still a symmetry, and if so in what sense?

Usually considered an "anomalous symmetry", not a normal symmetry, but not explicitly broken, different from spontaneously broken.

Two ideas:

Identify the reason behind counterexamples of Noether's theorem in QFT (theories with a continuous symmetry and no Noether current): the symmetry mixes Haag duality violating sectors. Some simple classification of cases.

ABJ anomaly as an ordinary internal U(1) symmetry that, however, mixes the Haag duality violations of the theory. This clarifies and unifies the origin of its main features in a perspective based on ordinary symmetry ideas:

1) It is a continuous symmetry but does not have a conserved Noether current

2) Goldstone theorem still applies (pions)

3) Anomaly quantization: general anomaly coefficient is proportional to an integer: calculation of possible anomaly coefficients

4) Anomaly matching (the coefficient of the anomaly matches between the UV and IR)

Operator algebras and regions

QFT: R $\longrightarrow$ $\mathcal{A}(R)$ (net of von Neumann algebras)

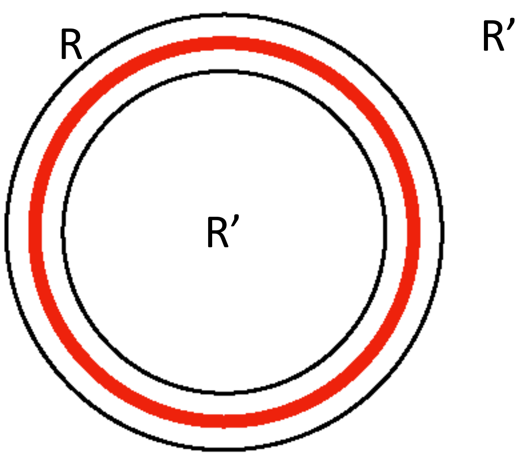
Two different meanings of the world "local":

1) $\mathcal{A}(R) \equiv \bigvee_{B\,\mathrm{ball}\,, \cup B = R} \mathcal{A}(B)$ Additivity: operators constructed from local degrees of freedom

2) $\mathcal{A}(R) \subseteq (\mathcal{A}(R'))'$ Causality.

Non local operators

It is "localized" in R: commutes with local operators in R'
But cannot be additively produced by fields in R



''Complete theory''

The local observable algebras formed additively with local degrees of freedom are the maximal ones compatible with causality.

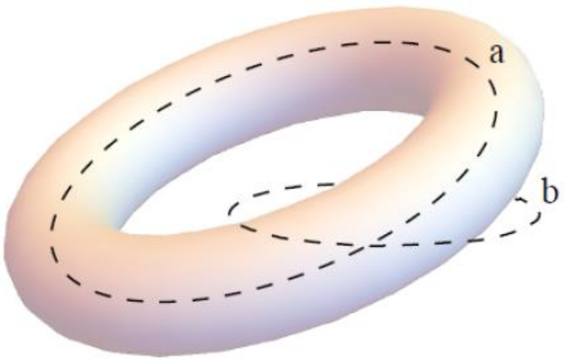
$\mathcal{A}(R) = (\mathcal{A}(R'))'$ Haag duality

$\mathcal{A}_{\max}(R) \equiv (\mathcal{A}(R'))'$ maximal algebra

Non complete theories: When there are non-local operators they always come in dual pairs:

$$\left.\begin{aligned} \mathcal{A}_{\max}(R) &= \mathcal{A}(R) \vee \{a\} \\ \mathcal{A}_{\max}(R') &= \mathcal{A}(R') \vee \{b\} \end{aligned}\right\}$$

a,b, dual ''non local'' operators.

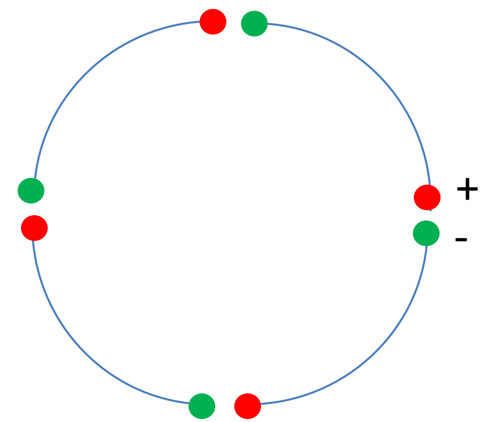


Not all a's and b's commute to each other

"Non local operator" is a relative notion. An operator (Wilson loop) can be non local in a region (ring) and additive on a different region (ball that contains the ring). Expected Haag duality for balls.

Example: Regions with non trivial homotopy groups π_1 or π_{d-3}

Gauge theories: Wilson loops $W_r \equiv \mathrm{Tr}_r \mathcal{P}\, e^{i \oint_C dx^\mu A^r_\mu}$



Not all Wilson loops are non locally generated in a ring-like region: for charged representations the loops can be broken in Wilson lines ended in charged operators

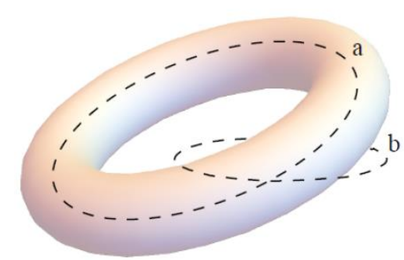
$$\phi_r(x)\, P e^{i \int_x^y dx^\sigma A_\sigma}\, \phi_r^\dagger(y)$$

Dual operators are the 't Hooft loops

For the free Maxwell field these are exponentials of the magnetic and electric fluxes for any charges

$W_q = e^{i\,q\,\Phi_B}$ $\quad T_q = e^{i\,q\,\Phi_E}$ gives $A = \mathbb{R}\,, B = \mathbb{R}\,, d \neq 4\,,$ $\quad A = \mathbb{R}^2\,, B = \mathbb{R}^2\,, d = 4$

$$W_q\, W_{q'} = W_{q+q'}\,, \quad T_g\, T_{g'} = T_{g+g'} \qquad W_q\, T_g = e^{i\,q\,q}\, T_g\, W_q$$



For QED: $A = U(1)\,, B = Z\,, d \neq 4\,,$ $\quad A = U(1)\times\mathbb{Z}\,, B = \mathbb{Z}\times U(1)\,, d = 4$

Global symmetry + HDV: obstructions to the strong form of Noether theorem

V. Benedetti, H.C., J. Magan (2022)

Non local classes can be charged or uncharged under the global symmetry

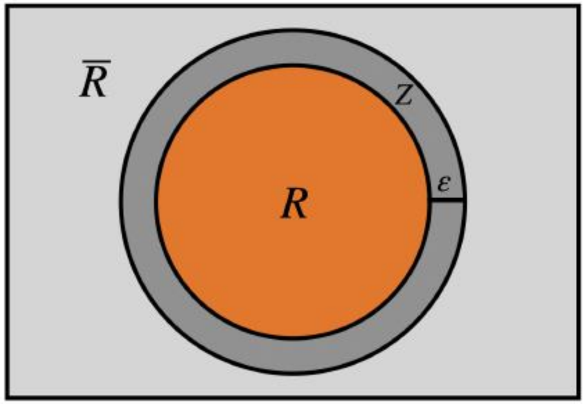
$$U(g)\,\mathcal{A}(R)\,U(g)^{-1} = \mathcal{A}(R) \longrightarrow U(g)\,\mathcal{A}(R)'\,U(g)^{-1} = \mathcal{A}(R)'\,, \qquad g \in G$$

$$U(g)\,\mathcal{A}_{\max}(R)\,U(g)^{-1} = \mathcal{A}_{\max}(R) \qquad A = \sum_{\lambda,s} O_{\lambda,s}\, a_\lambda\, \tilde{O}_{\lambda,s} \longrightarrow [a_\lambda]$$

$\longrightarrow$ Global symmetry acts as a point-like transformation of class labels $[a_\lambda] \to [a_{\lambda'}]$

Twists of the global symmetry: produces the transformation of U(g) in R and do not transform outside RZ

In general
- $\tau_g(R,Z) \in \mathcal{A}_{\max}(R\cup Z)$
- Transforms in $\mathcal{A}(R)$



A twist is additive if belongs to $\mathcal{A}(R\cup Z)$

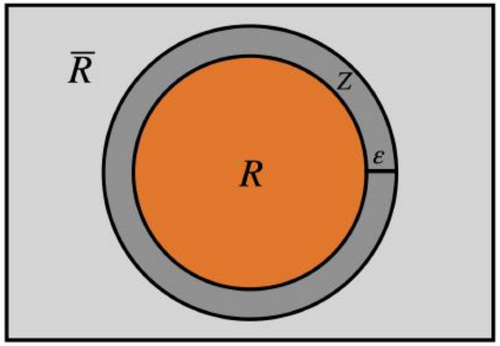
A twist is complete if transforms (as U(g)) in $\mathcal{A}_{\max}(R)$

Both types can always be constructed using modular theory

If a continuous symmetry has a Noether current there cannot be charged non-local classes

There are additive and complete twists for R

⟷ The non local classes of R are uncharged

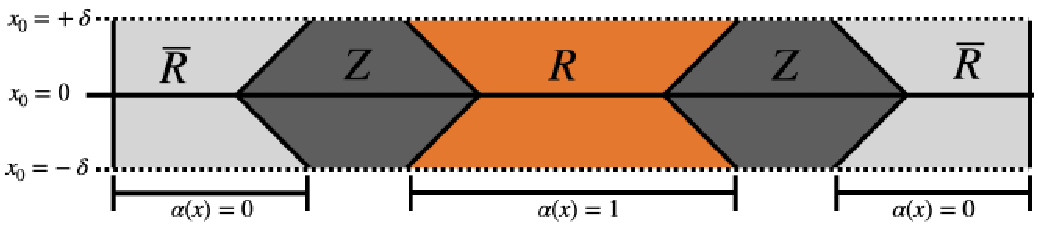


A Noether current allows the construction of additive charges:

$$\tau_\lambda(R,Z) = e^{i\lambda Q(R,Z)}\,, \qquad Q(R,Z) = \int d^dx\, \beta(x^0)\, \alpha(\vec{x})\, j_0(x)$$

The charge is also complete:

$$Q = Q(R,Z) + (Q - Q(R,Z))$$



Because Q generates the group transformation in any operator and (Q-Q(R,Z)) commutes with operators In $\mathcal{A}_{\max}(R)$ it must be that Q(R,Z) generates the full transformation in $\mathcal{A}_{\max}(R)$

Noether current implies no charged classes

If non local classes are charged twists cannot concatenate (as Noether charges would do)
(because of necessary boundary terms to eliminate gauge non invariance)

$$j^\mu = A^1_\nu\, F_2^{\nu\mu} - A^2_\nu\, F_1^{\nu\mu} \longrightarrow Q_R = \int d^3x\, f(x)\, J^0(x) = \int_{R\cup Z} d^3x\, f(x)\, [E^i_1(x)\, A^2_i(x) - E^i_2(x)\, A^1_i(x)]$$

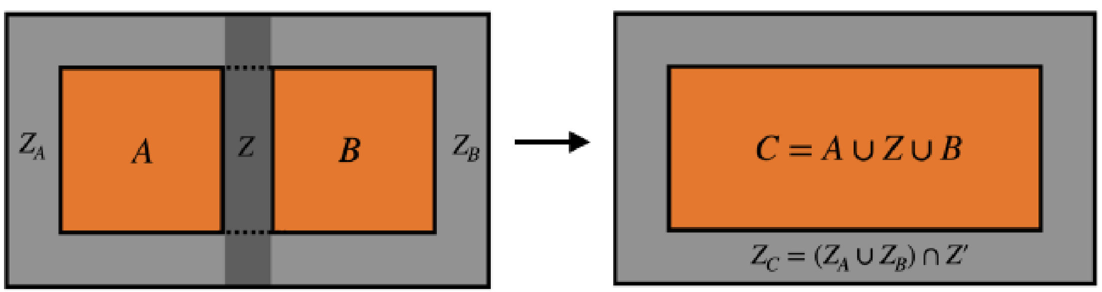
Non gauge invariance localized in Z can be cured

$$C = \int d^3x\, \partial_i f(x)\, [E^1_i(x)\, I_2(x) - E^2_i(x)\, I_1(x)] \qquad I_a(x) = -\int_Z d^3y\, \partial^y_i v(x,y)\, A^a_i(y)$$

$$\begin{aligned} \nabla^2_y\, v(x,y) &= \delta(x-y)\,, \qquad && x,y \in Z\,, \\ n^i\, \partial^y_i\, v(x,y) &= 0\,, && y \in \partial Z \end{aligned}$$

$$\tilde{Q}_R = Q_R + C$$

The charge does not have canonical dimensions A -> B



Some examples of violations of the Noether's theorem:

All known examples have charged non local sectors

Free graviton
No stress tensor.
Non local classes have Lorentz indices
Poincare symmetry mixes classes.

Two free Maxwell fields
No current for rotation symmetry.
Symmetry mixes Wilson loops.

Duality symmetry Maxwell field
No duality current
Symmetry mixes electric and magnetic fluxes.

Maxwell field for dimension $d \neq 4$
Derivatives of free scalar $d>2$
No dilatation current.
Symmetry mixes classes with dimensionful labels.

Free and massless.

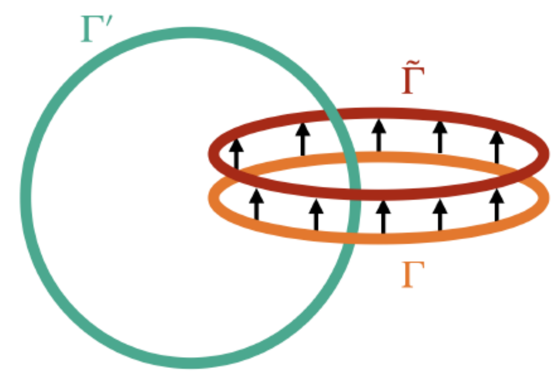
ABJ anomaly $\longrightarrow$ Interacting case?

An interpretation of Weinberg-Witten theorem:

V. Benedetti, H.C, J. Magan (2021)

"There is no Noether current corresponding to a charged particle with helicity h=1 or greater, and there is no stress tensor for a theory with a massless particle with helicity h=2 or greater"

A (free) graviton in d=4 has closed two forms:



$$
\begin{aligned}
A_{\mu\nu} &= R_{(\mu\nu)(\alpha\beta)}\, a^{\alpha\beta}\,, \qquad a^{\alpha\beta} = -a^{\beta\alpha}\,,\\
B_{\mu\nu} &= R_{(\mu\nu)(\alpha\beta)}\, (x^\alpha b^\beta - x^\beta b^\alpha)\,,\\
C_{\mu\nu} &= R_{(\mu\nu)(\alpha\beta)}\, c^{\alpha\beta\gamma} x_\gamma\,, \qquad c^{\alpha\beta\gamma} = -c^{\beta\alpha\gamma} = -c^{\alpha\gamma\beta}\,,\\
D_{\mu\nu} &= R_{(\mu\nu)(\alpha\beta)}\, (x^\alpha d^{\beta\gamma} x^\gamma - x^\beta d^{\alpha\gamma} x^\gamma + \frac{1}{2} d^{\alpha\beta} x^2)\,, \qquad d^{\alpha\beta} = -d^{\beta\alpha}
\end{aligned}
$$

$$O_\Gamma = \int_\Sigma d\sigma^{\mu\nu}\, R_{\mu\nu\alpha\beta}(x)\, f^{\alpha\beta}(x)$$

$$f^{\alpha\beta}(x) = a^{\alpha\beta} + (x^\alpha b^\beta - x^\beta b^\alpha) + c^{\alpha\beta\gamma} x_\gamma + (x^\alpha d^{\beta\gamma} x^\gamma - x^\beta d^{\alpha\gamma} x^\gamma + \frac{1}{2} d^{\alpha\beta} x^2)$$

$$[O_\Gamma, O_{\Gamma'}] = i\, \left(a \cdot \cdot \tilde{d}^* + 2\, b \cdot \tilde{c}^* - 2\, c^* \cdot \tilde{b} - d^* \cdot \cdot \tilde{a} \right)$$

There is a $\mathbb{R}^{20}$ group of generalized symmetries. The charges have Lorentz indices and transform non trivially as a linear representation of Poincare, conformal, and duality symmetries.

→ no stress tensor, no dilatation current, no duality current

A Noether current cannot exist when there are charged classes (of any topology) for the continuous symmet

This scenario is quite rare, why?

Classes and dual classes non invariant under a continuous symmetry: both must form a continuum.

What type of actions of a continuous 1-parameter symmetry on non local classes?

Let us assume Abelian classes .

"One dimensional case": fusion $a_1 + a_2$ $b_1 + b_2$ Commutation relations $a\, b = e^{i\, a\, b}\, b\, a$

$\longrightarrow$ $a(\lambda) = e^{\lambda}\, a$ $b \to e^{-\lambda}\, b$

Dual classes are non-compact and continuous group R. (Example: dilatation symmetry for free Maxwell field d≠4)

"Two dimensional cases": $a = (a_1, a_2)$ $b = (b_1, b_2)$ $a\, b = b\, a\, e^{ia\cdot b}$

$\longrightarrow$ $a \to M(\lambda)\, a\,,$ $b \to (M(\lambda)^T)^{-1}\, b$

Apart from dilatations, there is rotation group U(1), for non compact sectors $A = \mathbb{R} \times \mathbb{R}$ $B = \mathbb{R} \times \mathbb{R}$

(Example: rotation between two free Maxwell fields, duality for Maxwell field)

Reason for free massless models:

V. Benedetti, H.C, J. Magán (2022)

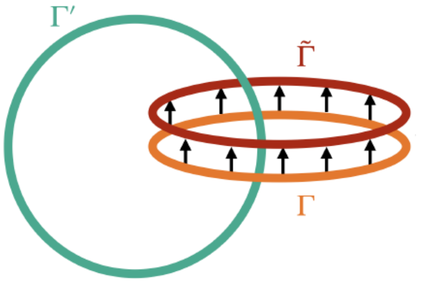
Most of the cases give non-compact sectors: continuous dual sectors, non commuting to each other. Non compact generalized symmetries (independently of any global symmetry acting on it) generally leads to free massless theories:

Both are continuous and considering the charge generators as conserved fluxes of form fields

$$\Phi_F = \int_{\Sigma_F} F\,, \qquad \Phi_G = \int_{\Sigma_G} G$$

$$[\Phi_F, \Phi_G] = i$$

commutator must be a c-number



The cross correlator is necessarily a "linking number term" for the two closed form-fields, a massless term that cannot renormalize

$$\langle F(x)G(0)\rangle \;=\; \int \frac{d^d p}{(2\pi)^{d-1}}\,\theta(p^0)\,\delta(p^2)\,e^{ipx}\,(P^{(k)}\tilde{*})(p) \qquad \Box\,\langle F(x)G(0)\rangle = 0$$

A more detailed analysis gives a free massless sector

However, among the two-dimensional cases: there is also a unique possibility that allows for compact sectors and interacting models, the “ABJ anomalous case”

$A = \mathbb{Z} \times U(1) \quad B = U(1) \times \mathbb{Z}.$ (as in QED in d=4)

$$a_1, b_2 \in \mathbb{Z}\,, \qquad a_2 \equiv a_2 + 2\pi\,,\, b_1 \equiv b_1 + 2\pi$$

$$(a_1, a_2) \to (a_1, a_2 + \lambda\, a_1)\,, \qquad M(\lambda) = \begin{pmatrix} 1 & 0 \\ \lambda & 1 \end{pmatrix}$$

$$(b_1, b_2) \to (b_1 - \lambda\, b_2, b_2)$$

In this case we would have:

1) A continuous global U(1) symmetry without Noether current (it changes classes)

2) Goldstone theorem: it follows from existence of twists operators (weak Noether theorem) and does not require Noether’s current (Buchholz, Doplicher, Longo, Roberts (1992))

3) “Quantization” of the group action: the compatibility of the cycle of the symmetry group $\lambda \equiv \lambda + \lambda_0$ with the one of the U(1) non-local operators im $\lambda_0 = 2\pi\, n$ (anomaly quantization)

4) If the non local operators exist in the IR limit, then the symmetry has to exist in the IR: there must be massless local excitations charged under the symmetry and the rates of group action and group of non local operators must match between different scales (anomaly matching).

Pion electrodynamics (IR effective model, anomaly seen at the classical level)

$$\mathcal{L} = \frac{1}{2}\partial_\mu \pi_0 \partial^\mu \pi_0 - \frac{1}{4\,e^2} F_{\mu\nu}F^{\mu\nu} + \frac{1}{8\mu}\,\epsilon_{\mu\nu\rho\sigma}\pi_0 F^{\mu\nu}F^{\rho\sigma}$$

$$J^\mu = \mu\,\partial^\mu \pi_0\,, \qquad \partial_\mu J^\mu = \frac{1}{8}\,\epsilon_{\mu\nu\rho\sigma}F^{\mu\nu}F^{\rho\sigma}$$ anomaly equation

$$\tilde{J}^\mu = \mu\,\partial^\mu \pi_0 - \frac{1}{2}\,\tilde{F}^{\mu\nu}A_\nu\,, \quad \partial_\mu \tilde{J}^\mu = 0$$ New, conserved, but non gauge invariant current

$$\tilde{Q} = \int d^3x\, \tilde{J}^0(x) = \int d^3x\, \left(\mu\,\dot{\pi}_0(x) - \frac{1}{2}B^i(x)A_i(x)\right)$$ Corrected charge (gauge transformation is boundary term)

$$\begin{aligned}
\Big[B^i(x), E^j(y)\Big] &= i\,e^2\,\epsilon^{ijk}\,\partial^x_k\,\delta(x-y)\,,\\
\Big[p_0(x), E^i(y)\Big] &= \frac{i\,e^2}{\mu}B^i(y)\,\delta(x-y)\,,\\
\Big[E^i(x), E^j(y)\Big] &= -\frac{i\,e^4}{\mu}\,\epsilon_{ijk}\left(\pi_0(y)\,\partial^k_y\delta(y-x) + \pi_0(x)\,\partial^k_x\delta(x-y)\right)
\end{aligned}$$ canonical commutation relations

$$\Big[\tilde{Q}, \pi_0(x)\Big] = -\mu\,i, \quad \Big[\tilde{Q}, B_i(x)\Big] = 0, \quad \Big[\tilde{Q}, p_0(x)\Big] = 0, \quad \Big[\tilde{Q}, E_i(x)\Big] = 0 \quad \Big[\tilde{Q}, T^{\mu\nu}\Big] = 0$$

$$U(\lambda)\,\pi_0(x)\,U(\lambda)^\dagger = \pi_0(x) + \lambda\,\mu\,, \qquad U(\lambda) = e^{i\lambda\tilde{Q}}\,.$$

The symmetry acts as an automorphism of the local algebras: only the pion is charged

What is special about this symmetry?
No (gauge invariant) Noether current: Are there non invariant non local classes?

$\partial_\nu \tilde{F}^{\mu\nu} = 0$ magnetic flux still conserved

$G^{\mu\nu} \equiv \frac{1}{e^2} F^{\mu\nu} - \frac{\pi_0}{\mu} \tilde{F}^{\mu\nu}\,, \quad \partial_\nu G^{\mu\nu} = 0$ redefined electric conserved flux: note it needs the pion

$\Phi_G = \int_\Sigma \star G\,, \quad \Phi_F = \int_{\tilde{\Sigma}} \star \tilde{F}$ $\qquad [\Phi_G, \Phi_F] = i$ flux commutator

$W_q = e^{iq\Phi_F}\,, \qquad T_g = e^{ig\Phi_G} \longrightarrow$ generic non local operator is a dyon (d=4) $D_{(g,q)}$

$D^R_{(g,q)}\, D^{R'}_{(g',q')} = e^{i\,(q\,g' - q'\,g)}\, D^{R'}_{(g',q')}\, D^R_{(g,q)}$ commutation relations

The action of the symmetry $[\tilde{Q}, \Phi_G] = i\,\Phi_F\,, \quad [\tilde{Q}, \Phi_F] = 0 \longrightarrow U(\lambda) D_{(g,q)} U^{-1}(\lambda) = D_{(g,q+\lambda\,g)}$

$$
\begin{aligned}
D^R_{(g_1,q_1+\lambda\,g_1)}\, D^R_{(g_2,q_2+\lambda\,g_2)} &= D^R_{(g_1+g_2,q_1+q_2+\lambda\,(g_1+g_2))}\,, \\
D^R_{(g,q+\lambda\,g)}\, D^{R'}_{(g',q'+\lambda\,g')} &= e^{i\,(q\,g'-q'\,g)}\, D^{R'}_{(g',q'+\lambda\,g')}\, D_{(g,q+\lambda\,g)}\,.
\end{aligned}
$$

respects the fusion rules
and commutation relations
Corresponds to the case discussed above

Since non local operators can be constructed additively in balls, this action on non local operators follows from the one on local operators. No freedom here.

Same story applies to massless QED: The ABJ anomaly (Adler, Bell, Jackiw1969)

$$S = \int d^4x \left[-\frac{1}{4e^2} F_{\mu\nu} F^{\mu\nu} + \overline{\psi}\, i \partial\!\!\!/ \,\psi - \overline{\psi}\, A\!\!\!/ \,\psi \right] \qquad J_5^\mu = \overline{\psi}\, \gamma^\mu \gamma^5\, \psi$$

$$\partial_\mu J_5^\mu = \frac{1}{16\pi^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma} \longrightarrow \tilde{J}_5^\mu = J_5^\mu - \frac{1}{4\pi^2} \tilde{F}^{\mu\nu} A_\nu \,, \qquad \partial_\mu \tilde{J}_5^\mu = 0$$

$$\tilde{Q} = \int d^3x \left[\psi^\dagger(x) \gamma^5 \psi(x) - \frac{B^i(x) A_i(x)}{4\pi^2} \right]$$

Taking into account Schwinger terms the new charge generates an internal U(1) symmetry

(Adler 1969)

$$\left[\tilde{Q}, A_i(x)\right] = 0 \,, \qquad \left[\tilde{Q}, E^i(x)\right] = 0 \qquad \left[\tilde{Q}, \bar{\psi}\left(\frac{1 \pm \gamma^5}{2}\right)\psi(x)\right] = \pm 2 \bar{\psi}(x) \left(\frac{1 \pm \gamma^5}{2}\right) \psi(x)$$

It is an internal U(1) symmetry

How does the anomalous chiral symmetry changes non local classes?: Witten's effect and quantization of the anomaly

$$\partial_\mu j^\mu = \frac{\beta}{4} F_{\mu\nu} \tilde{F}^{\mu\nu}$$

Take electric charges with minimal charge q_0 the (non local) WL charge have range $q \in [0, q_0)$

The non local TL have integer charges given by the Dirac quantization condition $g = \frac{2\pi}{q_0} k$

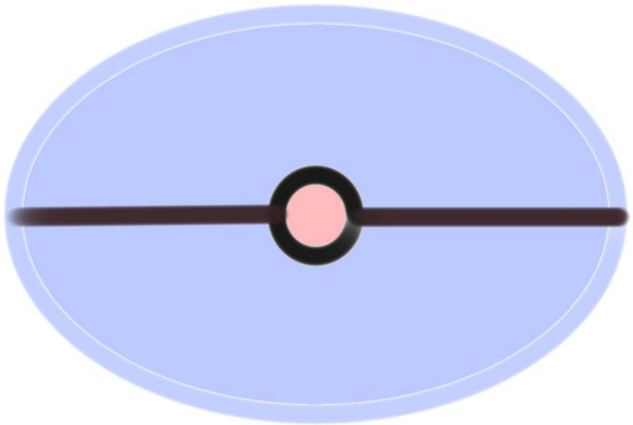
The current is normalized to have minimal charge equal to 1. Parameter $\lambda \in [0, 2\pi)$

Transformation with $\lambda(x)$ changes the action by $\delta S = \int \lambda(x)\, \partial_\mu j^\mu = \frac{\beta}{4} \int \lambda(x)\, F_{\mu\nu} \tilde{F}^{\mu\nu}$

$\longrightarrow$ $\nabla E = \beta \, (\nabla \lambda) \cdot B$

Monopole boundary conditions (TL) get mixed with
WL boundary conditions from classical equations of motion

$$(g, q) \to (g, q + \beta\, \delta\lambda\, g)$$



The possible values of the anomaly follow from compatibility of the group cycle and the group of non local operators

$\longrightarrow$ $\beta \times (2\pi) \times \left(\frac{2\pi}{q_0}\right) = n\, q_0$

$\longrightarrow$ $\partial_\mu j^\mu = n \frac{q_0^2}{16\,\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu}$

Coefficient of the anomaly without Feynmann diagrams

Remarks

The TL transforms non trivially because its expression in terms of local operators in a ball must contain gauge invariant chiral charged fields

QED then has the minimal possible value n=1 (given that the minimal gauge invariant chiral charge involves two electrons).

When gauge and chiral charge are the same it does not make sense: there is no gauge invariant local chirally charged operator, and hence no non-local operator can be charged.

Meaning of the integer n: The minimal charge of non local operators is n

$$\Psi_{k,m} = \int_0^{q_0} dq\, e^{i\,m\,\frac{2\pi\,q}{q_0}}\, D_{(\frac{k\,2\pi}{q_0},q)} \qquad\qquad U(\lambda)\,\Psi_{k,m}\,U(\lambda)^\dagger = e^{i\,n\,m\,k\,\frac{2\pi\,\lambda}{\lambda_0}}\,\Psi_{k,m}$$

The minimal charge of non local operators is n

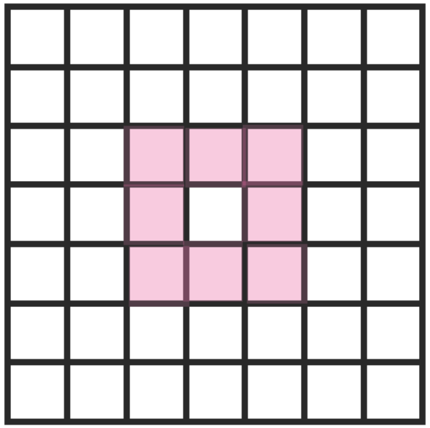
QCD has a spontaneously broken realization while in principle massless QED chiral symmetry is unbroken

Conjecture: (strong form of Noether's theorem): (assumptions? the split property plus time slice property…)

If there are no non local classes that are non invariant under a continuous global symmetry then there is a Noether current.

This is the converse of our previous statement . Requires UV completeness.

Heuristic Idea: there are complete additive charges for arbitrary partitions of the t=0 surface that should concatenate to the global operator. (Additive twists cannot concatenate when there are charged classes due to boundary terms. The same happens for splits).



Importance

General existence of currents except some controlled (mostly free massless) cases

No QFT with gravitons: Weinberg Witten → no stress tensor → charged Poincare classes at all scales → free decoupled gravitons

A more formal application: Algebraic QFT → fields - Wightman QFT

Final remarks

The anomalous action (U(1) that changes classes) required d=4 where we have magnetic and electric sectors in the same ring-like region, with the same topology. We can challenge that:

For d=2 QED (Schwinger model) the chiral symmetry is explicitly broken by instantons. There is no chiral symmetry. The same happens for currents with non abelian anomalies in d=4 (have discrete sectors).

Chiral symmetry in massive foton QED in d=4: now there is a Noether current $\tilde{J}_5^\mu = J_5^\mu - \frac{1}{4\pi^2}\tilde{F}^{\mu\nu}A_\nu\,, \qquad \partial_\mu \tilde{J}_5^\mu = 0\,.$

For d=6 we have an analogous to the pion electrodynamics

$$S = \frac{1}{2}\int d\pi_0 \wedge \star d\pi_0 + \frac{1}{2e^2}\int F \wedge \star F + \frac{1}{\mu}\int \pi_0\, F \wedge F \wedge F$$

$\pi_0 \to \pi_0 + \mathrm{cons}$ is an internal symmetry.

Again, this has a non gauge invariant current. However, TL and WL live in different topologies (fluxes of F and *F) and the anomalous action does not hold; the classes are invariant.
The conjecture is that this model cannot be UV completed.

Conjecture : An effective model with a continuous symmetry, no Noether current, and no non invariant HDV class cannot be UV completed.

There are several papers in the HEP literature that claim that this type of symmetry actions that mix with generalized symmetries are "non-invertible". However, the symmetry is a U(1) for the local physics. This invertibility is behind anomaly quantization.

The fact that the global internal symmetry is invertible (a group U(1)) is in accordance with the DHR theorem:
It can be rephrased as "All 0-form symmetries for d>2 come from a group".