Entropic order parameters:

Two algebras for the same region → two states for the same algebra → relative entropy

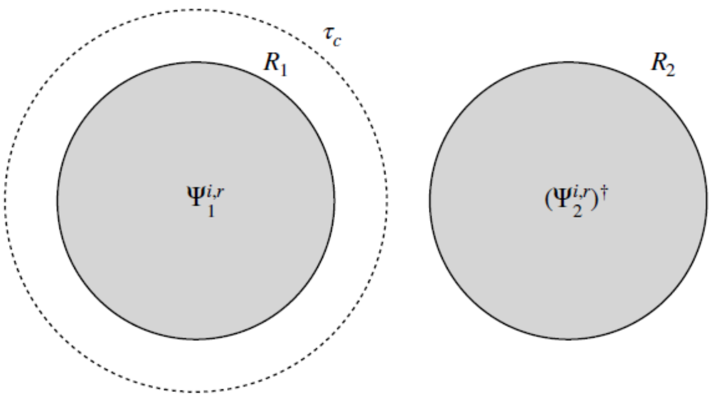
Conditional expectation: linear positive map (chanel) from an algebra to a subalgebra that keeps the subalgebra invariant $\varepsilon : \mathcal{A}_{\text{max}} \to \mathcal{A}_{\text{add}}$

Uses: lift a state from the subalgebra to the algebra $\omega_{\mathcal{A}_{\text{add}}} \to \omega_{\mathcal{A}_{\text{add}}} \circ \varepsilon$

Conditional expectation in the case of group

$$m \in \mathcal{A}_{\text{max}}\,, \qquad \varepsilon(m) = \frac{1}{|G|} \sum_g \tau_g\, m\, \tau_g^{-1} \in \mathcal{A}_{\text{add}}$$

Entropic order parameter $S_{\mathcal{A}_{\text{max}}}(\omega|\omega \circ \varepsilon)$



Relative entropy between the vacuum and the vacuum with non local operators set to zero expectation value.

It is a measure of the statistics (expectation values) of non local operators, but has info of the additive algebra as well

It is function of the geometry of the region only

Standard non local operators determined only by geometry and the vacuum can also be defined

Complementarity diagram

$$\begin{array}{ccc} \mathcal{A}_{\rm add}(R)\vee\{a\} & \xrightarrow{\ \varepsilon\ } & \mathcal{A}_{\rm add}(R) \\ \updownarrow{\prime} & & \updownarrow{\prime} \\ \mathcal{A}_{\rm add}(R') & \xleftarrow{\ \varepsilon'\ } & \mathcal{A}_{\rm add}(R')\vee\{b\} \end{array}$$

Dual conditional expectations are unique in QFT

For a pure global state we have the entropic certainty relation

$$S_{\mathcal{A}_{\rm max}(R)}(\omega|\omega\circ\varepsilon) + S_{\mathcal{A}_{\rm max}(R')}(\omega|\omega\circ\varepsilon') = \log\lambda \qquad (\log|G|)$$

Index of the inclusion $\mathcal{A}_{\rm add}(R) \subset \mathcal{A}_{\rm max}(R)$

H. Casini, M. Huerta, J. Magan, D. Pontello e-Print: 1905.10487 [hep-th]

J.Magan, D. Pontello e-Print: 2005.01760 [hep-th]

S. Hollands e-Print: 2009.05024 [quant-ph]

Feng Xu e-Print: 1812.01119 [math-ph]

$\mathcal{B} \subset \mathcal{A}$

Conditional expectation

$\epsilon : \mathcal{A} \to \mathcal{B}$

$$\varepsilon(m_+) \geq Ind(\varepsilon)^{-1} m_+$$

Entropic certainty and uncertainty relations

$$S_{\mathcal{A}_{\max}(R)}(\omega|\omega\circ\varepsilon)+S_{\mathcal{A}_{\max}(R')}(\omega|\omega\circ\varepsilon')=\log|G|$$

$\longrightarrow$ $$S_{\mathcal{A}_{\rm add}(R)\vee\{a\}}\left(\omega|\omega\circ\varepsilon\right)\leq\log|G|\,,\qquad S_{\mathcal{A}_{\rm add}(R')\vee\{b\}}\left(\omega|\omega\circ\varepsilon'\right)\leq\log|G|$$

But they cannot saturate at the same time: uncertainty relations for non commuting opertators a,b

$$\tau^2=1\,,\quad\psi^2=1\,,\quad\tau\,\psi=-\psi\,\tau$$

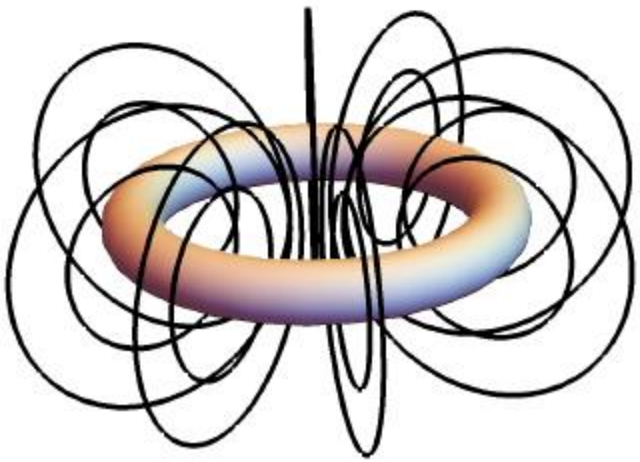
Z2 group

$$1-|\langle\tau\rangle|^2-|\langle\psi\rangle|^2\geq 0$$

It contains information on the rest of the algebra (improvements by additive operators)

It gives a handle in computations

$$S_{\{a\}}\left(\omega|\omega\circ\varepsilon\right)\leq S_{\mathcal{A}_{\rm add}(R)\vee\{a\}}\left(\omega|\omega\circ\varepsilon\right)\leq\log|G|-S_{\{b\}}\left(\omega|\omega\circ\varepsilon'\right)$$



Heuristics of area law:

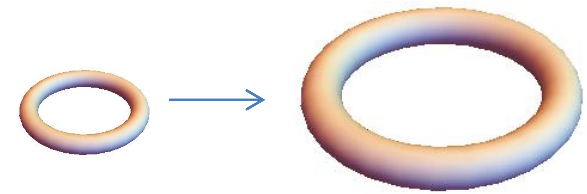
an area worth of decoupled loops with approx constant expectation value

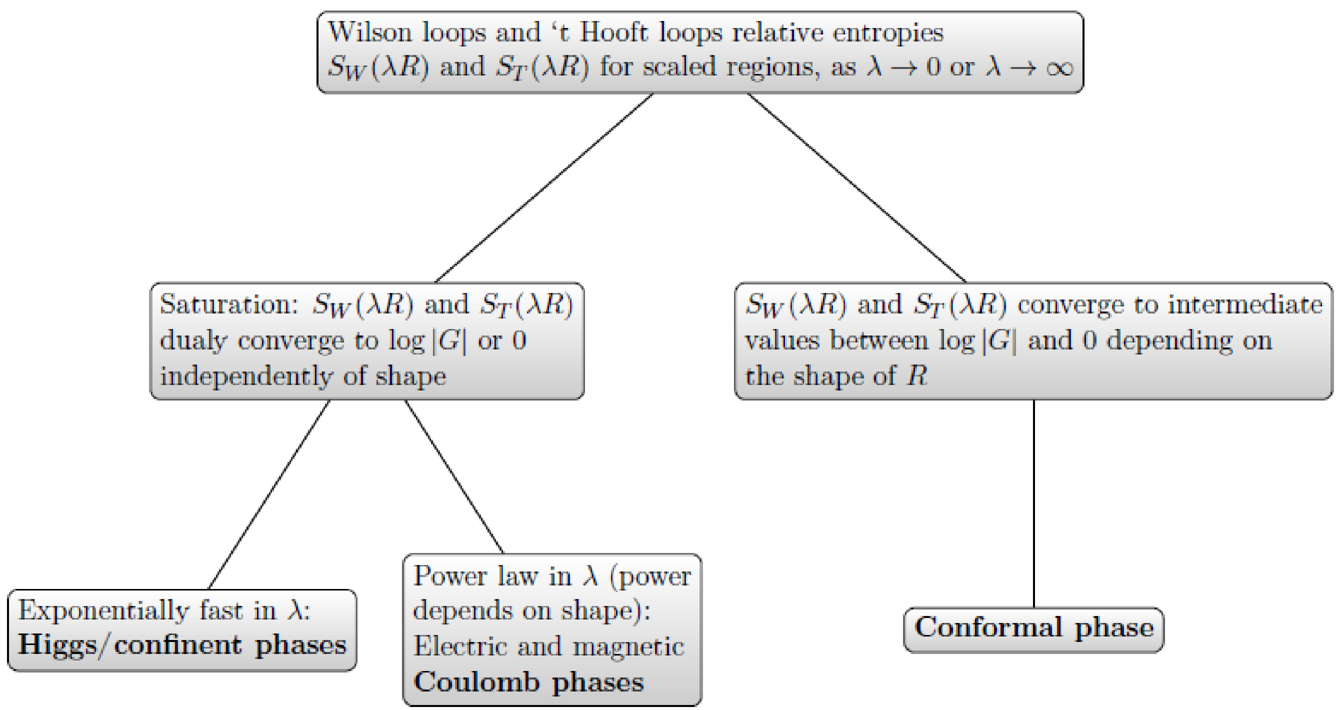
→ the relative entropy for the exterior of the orange ring approaches Log|G| exponentially in the number of loops (then exponentially in the area)

→ The relative entropy in the ring has an area law because of the certainty relation

Non abelian gauge phases.
Scaling order parameters.

Fat rings. Dual of an area law is a constant law:
Matches the same story for ordinary symmetries:
Massive and SSB cases.





Modular invariance as completeness

Plan of the talk:

1) Modular invariance (S transformation) as completeness of d=2 CFT's (through Renyi entropies)

2) Measuring the amount of incompleteness: the global index

3) Application to selection rules for the RG of minimal models

4) Some infinite index cases

5) Some results in higher dimensions

Modular invariance as completeness

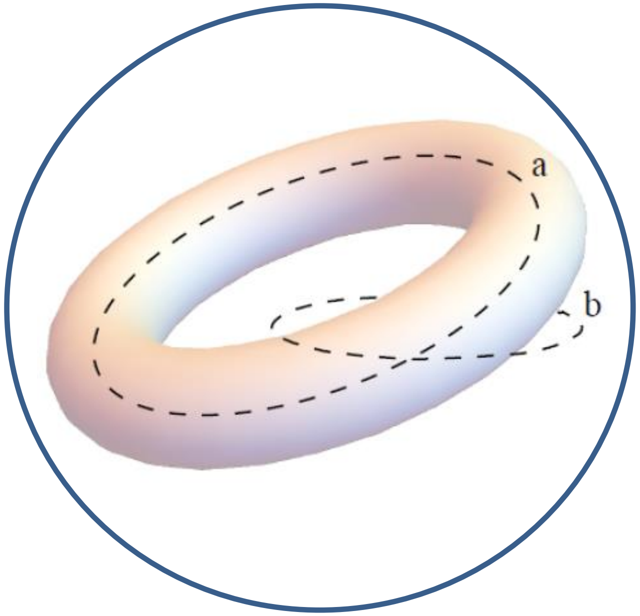
Based on work with V. Benedetti, Y. Kawahigashi, R. Longo, J. Magán (2024)

Completeness : Algebras generated by field operators are the maximal ones compatible with causality

$$\mathcal{A}_{\max}(R) = \mathcal{A}(R) = (\mathcal{A}(R'))'$$ Haag duality (for the additive algebra)

Remark:

"Non local operator" is a notion relative to a topologically non trivial region. An operator (Wilson loop, t' Hooft loop) can be non local in a region (ring) and is always additive in the ball that contains the ring (in general there is Haag duality for balls).



As all operators are ultimatelly generated by local ones the description of a CFT in terms of primary fields and boostrap data does not need any addition to describe HDV/GS.

$$(\Delta_i, s_i) \qquad f_{ijk}$$

Question: what is the imprint of Haag duality violations / generalized symmetries in the boostrap data?

For example: how to diagnose from the CFT data if a given model can be extended or is the neutral part of another one under a symmetry group? (so it has two-ball HDV)

Question: what is the imprint of Haag duality violations / generalized symmetries in the boostrap data?

For d=2 CFT's the question can be answered:

(S) Modular invariance ⟷ Completeness

Modular invariance usually invoked as an axiom. or even by "unitarity", but there are perfectly good models (as relativistic quantum models) that are non modular invariant.

How to make the connection?

Algebraic literature: proposed by Rehren (2001). Unpublished proofs by R. Longo, Y. Kawashigashi, and M. Muger (2004).

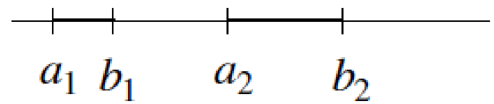
Renyi entropies: connection between partition functions through the replica trick to local algebras.

Renyi entropy

$$S_n = (1-n)^{-1} \log \mathrm{tr} \rho^n$$

For one interval $S_n(R) = \frac{(n+1)\,c}{6n} \log(r/\varepsilon)$

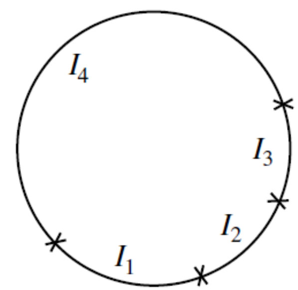
For two intervals: Renyi mutual information



$$x = \frac{(b_1-a_1)(b_2-a_2)}{(a_2-a_1)(b_2-b_1)} \in (0,1)$$

$$I_n(R_1, R_2) = S_n(R_1) + S_n(R_2) - S_n(R_1 \cup R_2)$$

$$I_n(x) = -\frac{(n+1)\,c}{6n} \log(1-x) + U_n(x)$$



If $S_n(I_1 \cup I_3) = S_n(I_2 \cup I_4)$ then $U_n(x) = U_n(1-x)$

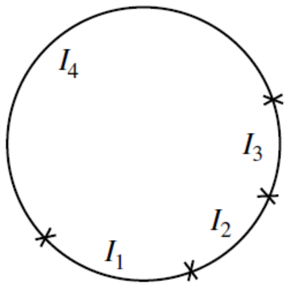
However, the entropies for a global pure state are equal for commutant algebras $S_n(\mathcal{A}) = S_n(\mathcal{A}')$ rather tan complementary regions.

Then $U_n(x) = U_n(1-x)$ ⟷ Haag duality for two intervals (completeness)

Example with non symmetric U: chiral scalar (the neutral part of a chiral complex fermion under the U(1) charge symmetry)

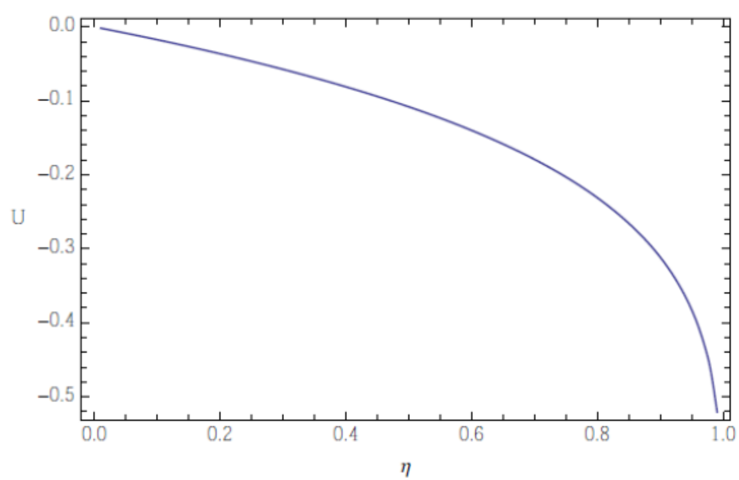
$$j(x^+) = \partial_+ \phi \qquad H = \frac{1}{2}\int dx\, j(x)^2\,, \quad [j(x), j(y)] = i\delta'(x-y)$$

$$O_{13} = \phi(x_1) - \phi(x_3) = \int_{x_1}^{x_3} dx\, \partial_x \phi(x)$$



$$[O_{13}, O_{24}] = i$$

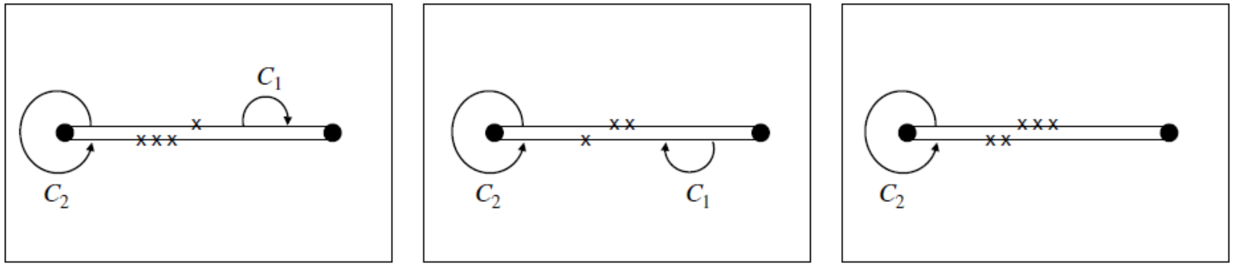
$$(\mathcal{A}_{\rm add}(I_1 I_3))' = (\mathcal{A}(I_1) \vee \mathcal{A}(I_3))' = \mathcal{A}(I_2) \vee \mathcal{A}(I_4) \vee O_{24} = \mathcal{A}_{\rm add}(I_2 I_4) \vee O_{24}\,;$$
$$(\mathcal{A}_{\rm add}(I_2 I_4))' = (\mathcal{A}(I_2) \vee \mathcal{A}(I_4))' = \mathcal{A}(I_1) \vee \mathcal{A}(I_3) \vee O_{13} = \mathcal{A}_{\rm add}(I_1 I_3) \vee O_{13}$$



small x: large distances
x near 1: touching intervals

Replica trick

$$\mathrm{tr}\rho^n = \frac{Z[n]}{Z[1]^n}$$



For n=2 and two intervals the manifold has genus 1 and can be conformally mapped to a torus of radius 1 and hight l. Tranformation cost is universal and depends only on c (Liuville action) (Furukawa, Pasquier, Shiraishi 2009, Headrick 2010)

$$I_2(x) = \log Z[il] - \frac{c}{12} \log \left(\frac{2^8 (1-x)}{x^2} \right)$$

$$x = \left(\frac{\theta_2(il)}{\theta_3(il)} \right)^4, \quad 1-x = \left(\frac{\theta_2(i/l)}{\theta_3(i/l)} \right)^4, \quad l = \frac{{}_2F_1(\frac{1}{2},\frac{1}{2},1,1-x)}{{}_2F_1(\frac{1}{2},\frac{1}{2},1,x)}$$

$x \leftrightarrow 1-x$ ⟷ $l \leftrightarrow 1/l$

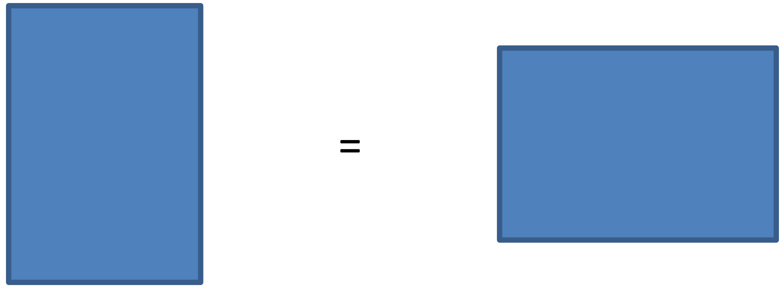
$$Z[\tau] = \mathrm{tr} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}, \; q = e^{i2\pi\tau} \qquad \tau = il$$

$$U_2(x) = \log Z[il] + \frac{c}{6} \log \left(\frac{x(1-x)}{2^4} \right)$$

The asymmetry is both a violation of modular invariance $Z[\tau] = Z[-1/\tau]$ and of Haag duality

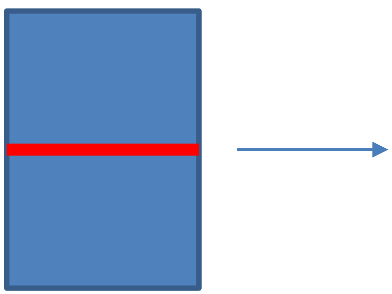
$$A_2(x) = U_2(x) - U_2(1-x) = \log Z[i\,l] - \log Z[i/l]$$

Why?



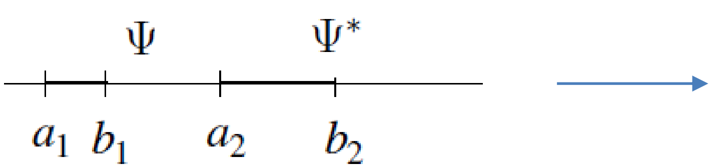
Complete model

Suppose we have a field charged under a group G and take the neutral algebra $\Psi, \Psi^* \to (\Psi\Psi^*)$



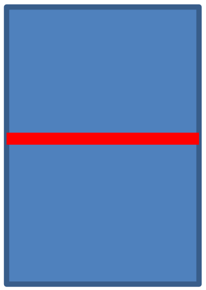
$$P = \frac{1}{|G|}\sum_{g\in G} U(g)$$

Projector onto the neutral states: not modular invariant anymore



Not an operator in the additive algebra of the intervals, yes an operator in the maximal algebra: no Haag duality for two intervals

The asymmetry is in general a complicated function of the cross ratio. But the limit x → 1 is “universal”. It is given by an index that only depends on the type of “symmetry”.



$$P = \frac{1}{|G|}\sum_{g\in G} U(g)$$

$$\frac{Z_{\mathcal{T}}(\beta)}{Z_{\mathcal{C}}(\beta)} = \langle P\rangle_\beta = \frac{1}{|G|}\sum_{g\in G}\langle U(g)\rangle_\beta$$

In the high temperature limit there are many independent charge fluctuations of all representations.

→ multiples of the regular representation

$$\lim_{\beta\to 0}\langle U(g)\rangle_\beta \to \delta_{g,1}$$

$$\lim_{\beta\to 0}\frac{Z_{\mathcal{T}}(\beta)}{Z_{\mathcal{C}}(\beta)} = \lim_{\beta\to 0}\langle P\rangle_\beta = \frac{1}{|G|}$$

$$A_2(x) = U_2(x) - U_2(1-x) = \log Z[i\,l] - \log Z[i/l]$$

$$U(0) = 0$$

$$U_2(1) = \lim_{l\to 0}\left(\log\frac{Z_{\mathcal{T}}[i\,l]}{Z_{\mathcal{C}}[i\,l]} - \log\frac{Z_{\mathcal{T}}[i/l]}{Z_{\mathcal{C}}[i/l]}\right) = \lim_{l\to 0}\log\frac{Z_{\mathcal{T}}[i\,l]}{Z_{\mathcal{C}}[i\,l]} = -\log|G|$$

In more generality the inclusion of the algebras $\mathcal{T} \subset \mathcal{C}$ has sectors or charges and non invertible symmetries

$$r \times s = \bigoplus_t N_{rs}^t \, t \quad \longrightarrow \quad N_{st}^{(r)} = N_{rt}^s \quad \longrightarrow$$ (non-integer) dimensions $(d_1, d_2, \cdots, d_n)$

Index of inclusion of algebras $\lambda = [\mathcal{C} : \mathcal{T}] = \sum n_r d_r$ $\qquad \left(p_r = \frac{n_r d_r}{\sum n_s \, d_s} \right)$

$$U_2(1) = \lim_{l \to 0} \left(\log \frac{Z_\mathcal{T}[i\,l]}{Z_\mathcal{C}[i\,l]} - \log \frac{Z_\mathcal{T}[i/l]}{Z_\mathcal{C}[i/l]} \right) = \lim_{l \to 0} \log \frac{Z_\mathcal{T}[i\,l]}{Z_\mathcal{C}[i\,l]} = -\log \lambda$$

This is the index of the inclusion of the model in a complete one. But one can also define an index of the inclusion between the maximal and additive algebras for two intervals $\mathcal{A}(R) \subseteq (\mathcal{A}(R'))'$
This is called the global index and is defined by the model itself without any completion.

$\mu_\mathcal{T} = \lambda^2 \, \mu_\mathcal{C}$ (Longo (1989)) $\longrightarrow$ $U_2(1) = -\frac{1}{2} \log \mu$

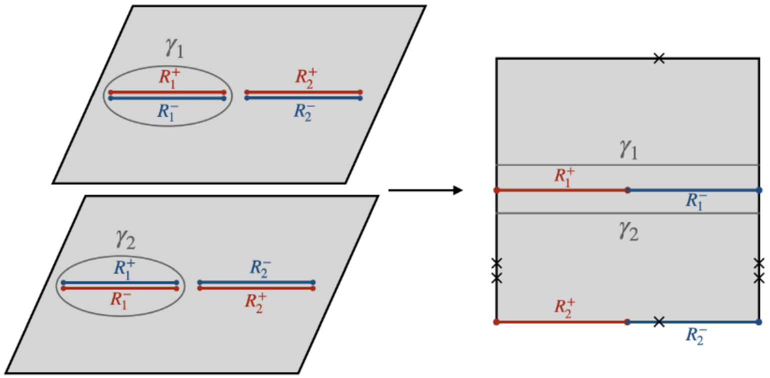
In summary:

$\log Z_\mathcal{C}[i/l] \simeq \frac{\pi\, c}{6l} + \text{exponentially small}\,, \quad l \ll 1$ $\longrightarrow$ Gap in the spectrum

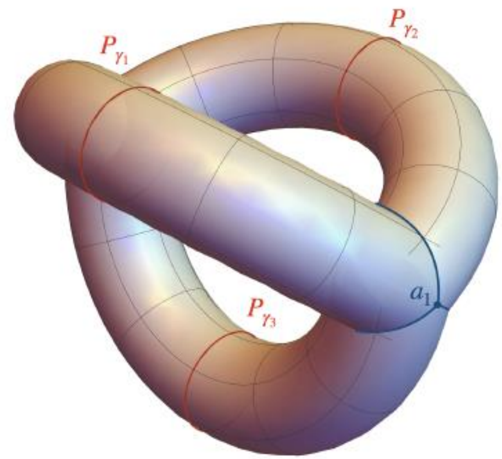
$\log Z_\mathcal{C}[il] \simeq \frac{\pi\, c}{6l} + \text{exponentially small}\,, \quad l \ll 1$ $\longrightarrow$ Modular invariance

$\log Z_\mathcal{T}[i\,l] = \frac{\pi\, c}{6l} - \frac{1}{2} \log \mu + \cdots\,, \quad l \ll 1\,.$ $\longrightarrow$ No modular invariance

Other Renyi entropies



n=2, two intervals



n=3, two intervals

$$S_n = (1-n)^{-1} \log \operatorname{tr} \rho^n$$

$$\lim_{x \to 1} U_n^-(x) = -\log \lambda$$

flat spectrum

How to compute the index: model defined as extension of local chiral algebra

$\chi_r(\tau) = \mathrm{Tr}_r\, e^{2\pi i \tau (L_0 - c/24)}$ Chiral characters

$\chi_r(-1/\tau) = \sum_s S_{rs}\, \chi_s(\tau)\,, \quad \chi_r(\tau+1) = \sum_s T_{rs}\, \chi_s(\tau)$ Modular transformation matrices

Model defined by the coupling matrix $Z = \sum M_{rs}\, \chi_r(\tau)\, \chi_s(\bar\tau)$ → $\varphi = \varphi_r \varphi_{\bar s}$

Modular invariance is $S\, M = M\, S\,, \quad T\, M = M\, T$

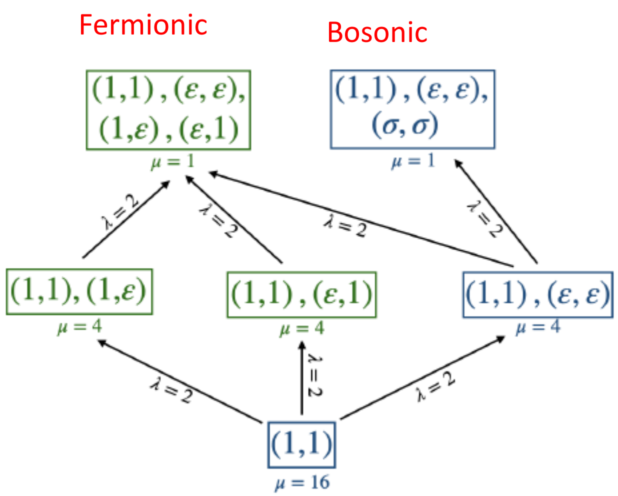
The T symmetry is equivalent to $h = \bar h + k\,, \quad k \in \mathbb{Z}$ necessary for causality $\langle \varphi(z)\varphi(0)\rangle = \frac{1}{z^{2h}\, \bar z^{2\bar h}}$

Using $\lim_{\tau\to 0} \frac{\chi_r(\tau)}{\chi_1(\tau)} = \frac{S_{0r}}{S_{00}} = d_r\,, \quad \lim_{\tau\to 0} \frac{\chi_1(\tau)}{\chi_1(-1/\tau)} = S_{00}$ $\mu = \sum_r d_r^2 = \tilde S_{00}^{-2}$

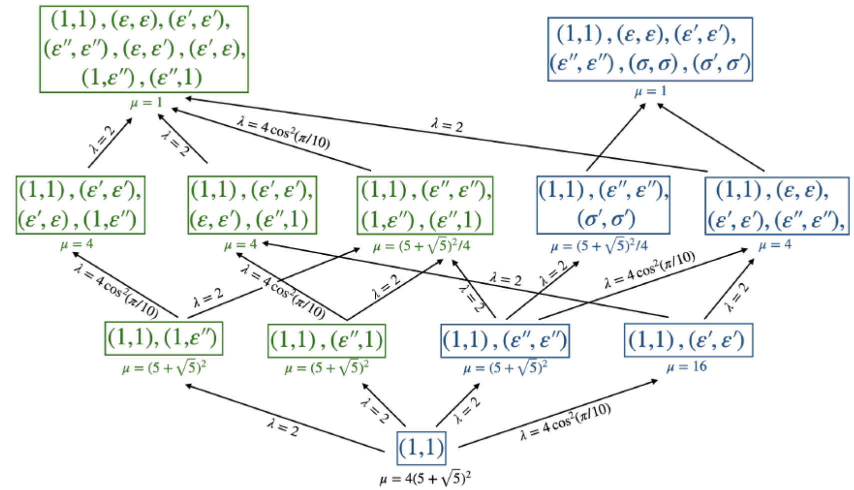
The global index is

$$\mu_T^{-1/2} = \lim_{l\to 0} \frac{Z(\tau)}{Z(-1/\tau)} = \left(\frac{\sum_i d_i^2}{\sum_{ij} d_i\, M_{ij}\, d_j} \right)^2$$

Some examples:



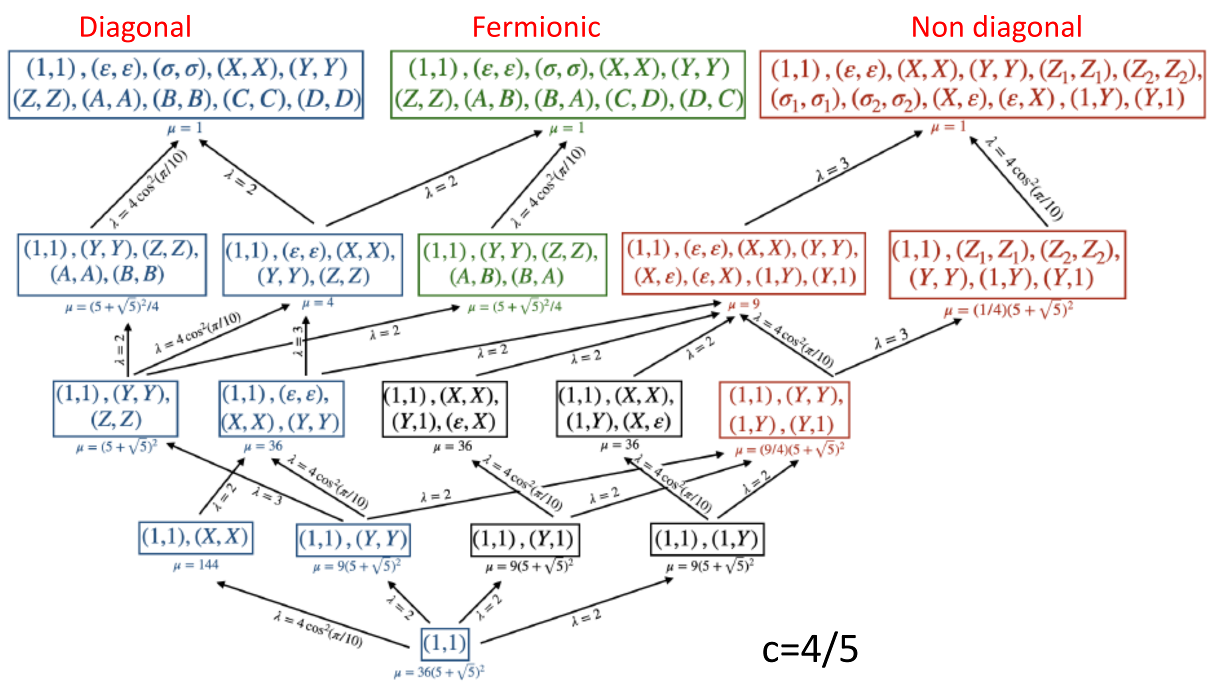
Ising model c=1/2



Tricritical Ising model c=7/10

In d=2 a model can have several inequivalent completions (and non group symmetry, "braided categories").

For d>2 the completion (for two l sectors) is unique and related to group symmetry.



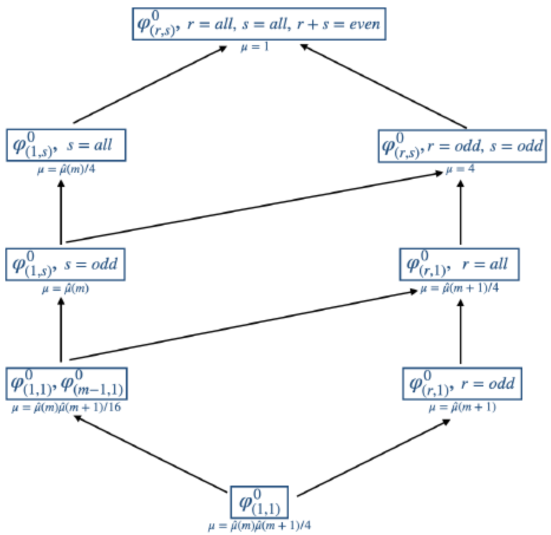
c=4/5

All subcategories classified for c<1 (Kawahigashi and Longo (2004))

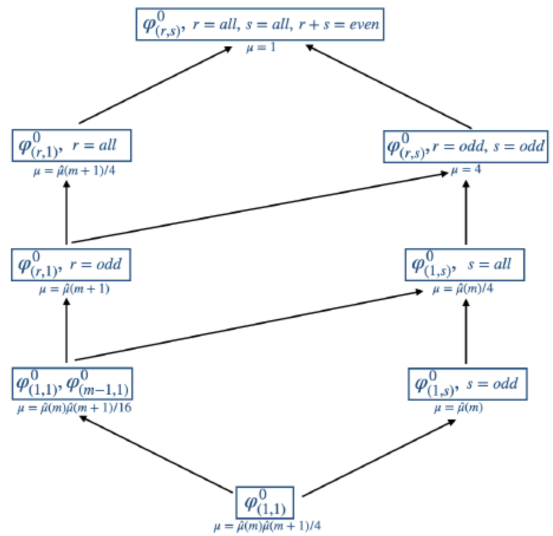
All submodels for c<1 (Benedetti, Casini, Magan (2024))

Minimal models

$$c = 1 - \frac{6}{m(m+1)}$$



odd $m \geq 5$



even $m \geq 6$

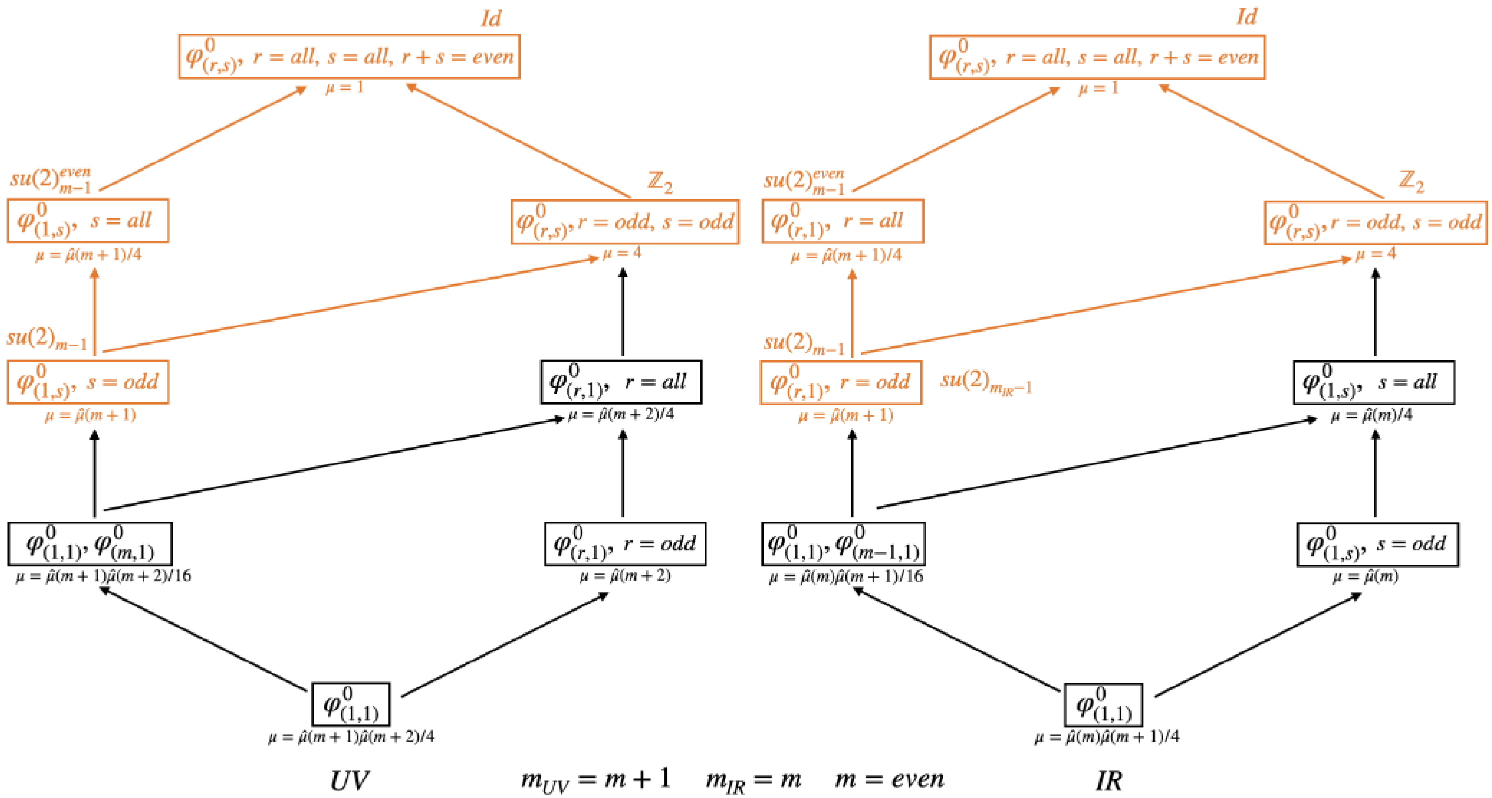
For general m there are 8 diagonal models

$$\hat{\mu}(m) = m^2 \sin^{-4}(\pi/m)/4.$$

	Fields included	Tensor Category	Global Index (μ)
1	$\varphi^0_{(r,s)}$ for all r and s	Id	1
2	$\varphi^0_{(r,s)}$ for r odd and s odd	$\mathbb{Z}_2$	4
3	$\varphi^0_{(1,s)}$ for all s	$su(2)^{\text{even}}_{m-2}$	$\frac{m^2}{16}\sin^{-4}\left(\frac{\pi}{m}\right)$
4	$\varphi^0_{(1,s)}$ for s odd	$su(2)_{m-2}$	$\frac{m^2}{4}\sin^{-4}\left(\frac{\pi}{m}\right)$
5	$\varphi^0_{(r,1)}$ for all r	$su(2)^{\text{even}}_{m-1}$	$\frac{(m+1)^2}{16}\sin^{-4}\left(\frac{\pi}{m+1}\right)$
6	$\varphi^0_{(r,1)}$ for r odd	$su(2)_{m-1}$	$\frac{(m+1)^2}{4}\sin^{-4}\left(\frac{\pi}{m+1}\right)$
7	$\varphi^0_{(1,1)}$ and $\varphi^0_{(m-1,1)}$	$su(2)^{\text{even}}_{m-2} \times su(2)^{\text{even}}_{m-1}$	$\frac{m^2(m+1)^2}{256}\sin^{-4}\left(\frac{\pi}{m}\right)\sin^{-4}\left(\frac{\pi}{m+1}\right)$
8	$\varphi^0_{(1,1)}$	(A_{m-1}, A_m)	$\frac{m^2(m+1)^2}{64}\sin^{-4}\left(\frac{\pi}{m}\right)\sin^{-4}\left(\frac{\pi}{m+1}\right)$

Application: Selection rules for RG of minimal models

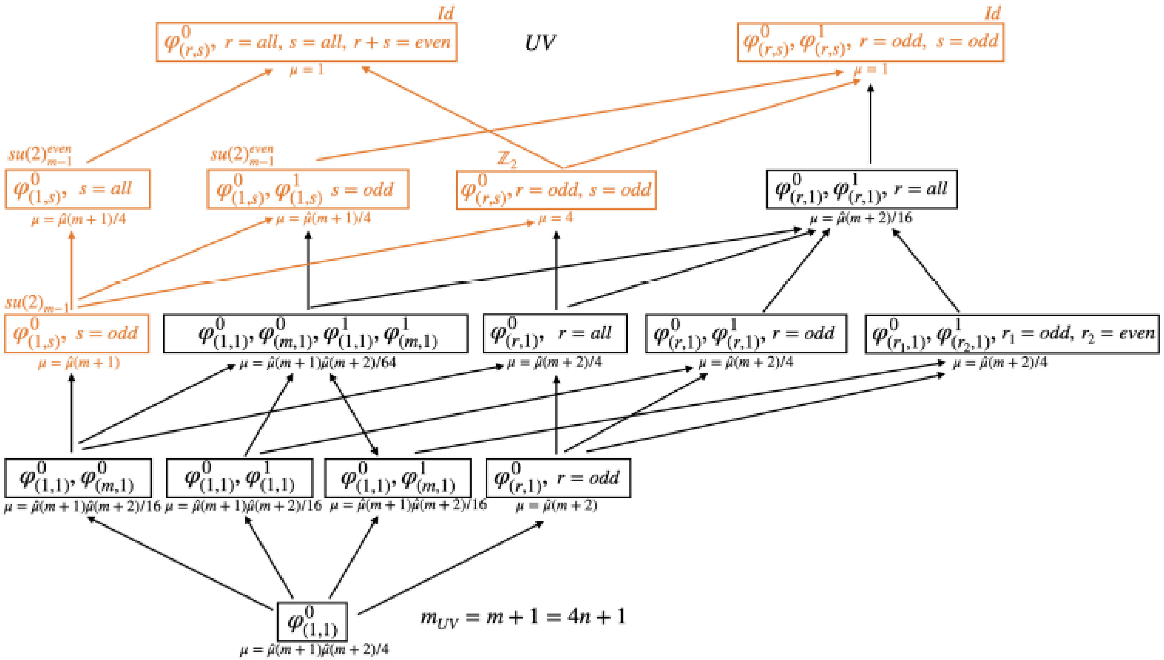
Benedetti, Casini, Magan 2024
C. Chang, Y. Lin, S. Shao, Y. Wang, X. Yin 2018
Y Nakayama, T. Tanaka 2024

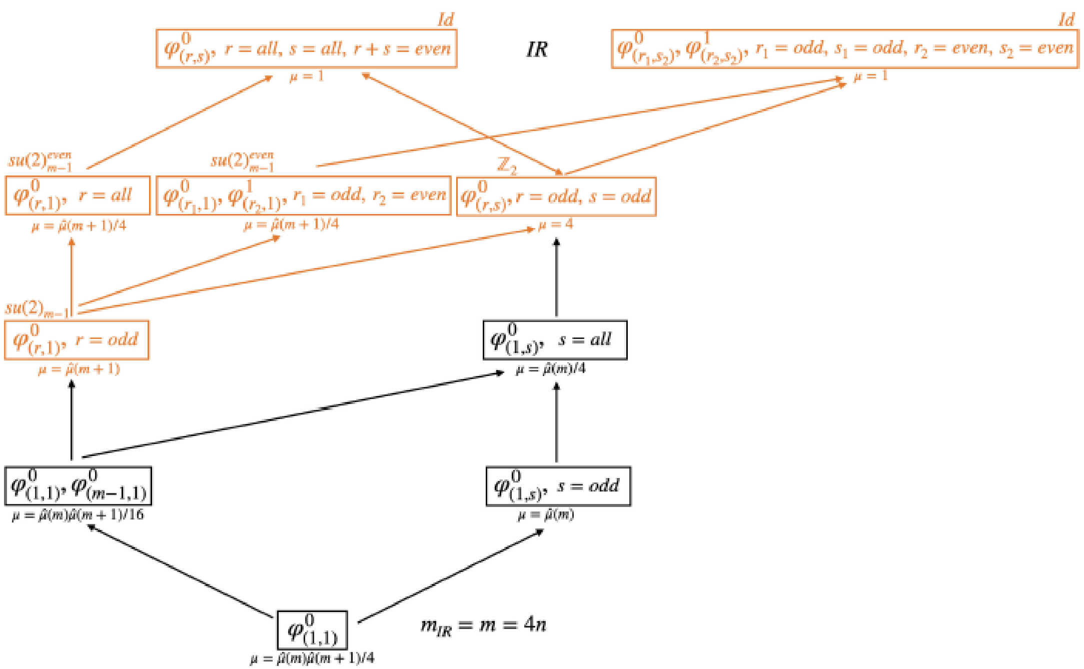


If the UV action is perturbed with a relevant scalar of the UV theory the flow can be thought to take place in the subalgebra generated by this field and the stress tensor.

The index and all structure of inclusions above this subalgebra are preserved by the RG. In the picture the flow started by the field (1,3) (Zamolodchikov's flow). The index is such that it is only matched by a submodel of $m_{IR} = m_{UV} - 1$ There are no other posible matchs for this flow (unless massive).
In general quite restricted set of posibilities: new ones for example (1,2) can go from m to m-1, (2,1) gapped.

Non diagonal extensions





Some cases of infinite index

If the relation between complete model and submodel is a Lie group $\tau = e^{i\, a_i\, L_i}$ $\qquad \langle \tau \rangle \sim e^{-k\, \frac{|\vec{a}|^2}{l}}$

$$\log \frac{Z_{\mathcal{T}}(\beta)}{Z_{\mathcal{C}}(\beta)} = \log \langle P \rangle_\beta \sim \frac{\mathcal{G}}{2} \log(\beta)$$

$$U_n(x) \sim -\frac{\mathcal{G}}{2} \log(-\log(1-x))\,, \quad x \to 1$$

H.C., M. Huerta, J. Magán, D. Pontello (2020)

Chiral scalar (U(1) fix point of the chiral complex fermion)

$$U_n(x) \sim -\frac{1}{2} \log(-\log(1-x))\,, \quad x \to 1$$

$$U_n(\eta) = \frac{\text{in}}{2(n-1)} \int_0^{+\infty} ds (\coth(\pi s n) - \coth(\pi s)) \times \log\left(\frac{{}_2F_1(1+is, -is; 1; \eta)}{{}_2F_1(1-is, is; 1; \eta)} \right).$$

Virasoro net for c=1

$$A_2(l) = \log\left(\frac{\chi_1(l)}{\chi_1(1/l)} \right) = \frac{1}{2} \log(l) + \log\left(\frac{1 - e^{-2\pi l}}{1 - e^{-2\pi/l}} \right) \longrightarrow U_2(x) \sim -\frac{3}{2} \log(-\log(1-x))$$

Virasoro net for c=1 fix point of the chiral current algebra $\hat{su}(2)_1 \;/\; SU(2)$

(Rehren 1993, R. Dijkgraaf, E. Verlinde, H. Verlinde 1988)

For the Virasoro net c>1 too many sectors to be a Lie group.

$$U_2(x) \sim \frac{(c-1)}{12} \log(1-x)$$

However, formula of Renyi 2 highly suspicious. No strong additivity!

$$\mathcal{A}(\gamma_1) \vee \mathcal{A}(\gamma_2) = \mathcal{A}(\gamma) \qquad P_{\gamma_1} P_{\gamma_2} \to P_\gamma$$

Higher dimensions

$$S^1 \times \cdots \times S^1 \qquad S^{d-1} \times S^1$$

→ No modular symmetry

Thermal effective action

Banerjee, J. Bhattacharya, S. Bhattacharyya, Jain, Minwalla, Sharma (2012)
Jensen, Kaminski, Koutun, Meyer, Ritz, Yarom (2012)
Benjamin, Lee, Ooguri, Simmons-Duffin (2023)

$$\log Z \simeq c_3 T^3 + c_1 T^1 + c_{-1} T^{-1} + \cdots$$ d=4 (even dimensions)

$$\longrightarrow \quad \log Z \simeq c_3 T^3 + c_1 T^1 - \log|G| + \cdots$$

In terms of the asymptotic density of primaries

$$\log \rho \simeq \log \rho_c - \log |G|$$

$$\log \rho \simeq 2\pi \sqrt{\frac{c\,\Delta}{6}} - \frac{3}{4}\log \Delta + \frac{1}{4}\log(c/96) - \log \lambda + \mathcal{O}(\Delta^{-1/2})$$

(for d=2 correction of Cardy formula gives the solution of original problem)

For Lie group symmetry (distinguishable in any dimension)

$$\log \rho \simeq \log \rho_c - \frac{\mathcal{G}\,(d-1)}{2\,d}\log(\Delta)$$

For example for d=4 logarithmic term has a universal form for a complete model proportional to

$$-4/3\,\log(\Delta)$$

Kang, Lee, Ooguri (2023)

Haag duality violations (or generalized symmetry) is lack of completeness.

Expression in terms of CFT data understood in d=2: violation of S modular invariance. Modular invariance is completeness for d=2 CFT's.

In higher dimension HDV for disconnected regions (associated to a symmetry group) detectable from the asymptotics of the density of states (at least for even d).

How to detect higher form symmetries in the CFT data?