Entanglement in Quantum Field Theory

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References

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  Notes On Some Entanglement Properties Of QFT

- ⋆ **I. Entanglement in finite systems**
- ⋆ **II. Entanglement in QFT**
- ⋆ **III. From the lattice to the continuum**
- ⋆ **IV. QFT from entanglement**

Part I: Entanglement in finite systems

Warm up: states and observables

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A general quantum state is represented by a unit-trace, positive semi-definite operator ρ acting on $\mathcal{H}$, aka **density matrix**.

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Entanglement $\iff$ non-separability of quantum systems

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When the global state is entangled, we necessarily lose information when we ignore one of the subsystems. The subsystems should be understood as forming a single inseparable entity...

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- Given a quantum system made of two subsystems A, B, a pure global state $|\psi\rangle$ is separable if we can write it as a tensor product, $|\psi\rangle = |\phi\rangle_A \otimes |\tilde{\phi}\rangle_B$. If not, it is entangled.

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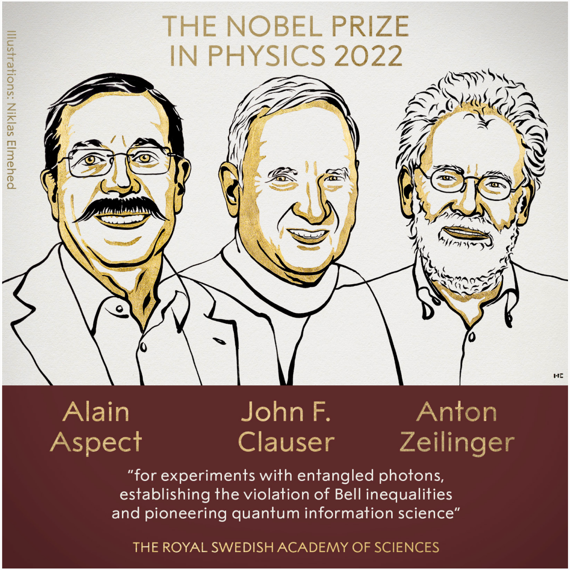
- Given a quantum system made of two subsystems A, B, a pure global state $|\psi\rangle$ is separable if we can write it as a tensor product, $|\psi\rangle = |\phi\rangle_A \otimes |\tilde{\phi}\rangle_B$. If not, it is entangled.
- If $|\psi\rangle$ is separable, the reduced density matrices on A and B are pure. If $|\psi\rangle$ is entangled, the reduced density matrices are mixed. In entangled states we necessarily lose information whenever we take a partial trace over one of the subsystems.

Entanglement is real...



A closer look?

- ⋆ Read about Bell inequalities, their relation to entanglement and the role of possible loopholes in the experimental tests (detection, locality, freedom of choice, etc.).

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An **algebra of operators** $\mathcal{A}$ is a set closed under linear combinations, operator products, and taking adjoints, which also includes scalar multiples of the identity,

$$1 \in \mathcal{A}, \quad a, b \in \mathcal{A}, \quad \alpha, \beta \in \mathbb{C} \quad \Rightarrow \quad \alpha a + \beta b \in \mathcal{A}\,, \quad ab \in \mathcal{A}\,, \quad a^\dagger \in \mathcal{A}$$

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Given some set of operators $\mathcal{A}$, we define its **commutant** $\mathcal{A}'$ as the set of operators which commutes with all the operators of $\mathcal{A}$,

$$\mathcal{A}' \equiv \{ b \,|\, [b, a] = 0, \forall a \in \mathcal{A} \}$$

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The smallest algebra which contains any set $\mathcal{A}$ is $\mathcal{A}''$.

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$$\omega(\alpha a + \beta b) = \alpha\omega(a) + \beta\omega(b)\,, \quad \omega(aa^\dagger) \geq 0\,, \quad \omega(1) = 1\,,$$

namely, it is linear, it produces a positive semi-definite result for operators with a positive spectrum and it is normalized.

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For any state ω acting on $\mathcal{A}$, $\exists!$ element of the algebra $\rho_{\omega,\mathcal{A}} \in \mathcal{A}$ such that

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Hence, a state acting on an algebra selects an operator in the algebra itself, the reduced density matrix.

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Given a global pure state ω, separability means that we can write

$$\omega = \omega_A \otimes \tilde{\omega}_B$$

where ω_A is a state on $\mathcal{A}_A$ and $\tilde{\omega}_B$ is a state on $\mathcal{A}_B$. Otherwise, ω is entangled.

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- Quantum observables organize themselves in algebras. A set of operators $\mathcal{A}$ is an algebra iff it coincides with its double commutant, $\mathcal{A} = \mathcal{A}''$.
- States take operators from the algebras and produce numbers out of them (expectation values).
- Any state ω acting on an algebra $\mathcal{A}$ selects a unique element of the algebra (the density matrix $\rho_{\omega,\mathcal{A}} \in \mathcal{A}$) such that the expectation value of any operator a of the algebra can be computed in the usual way, namely, as $\mathrm{tr}(\rho_{\omega,\mathcal{A}} a)$.

A closer look?

- ⋆ Read about the classification of von Neumann algebras and the contexts in which the different types appear in physics. [More in Witten's lectures]

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$$S(\rho_{\omega,\mathcal{A}}) = S(\rho_{\omega,\mathcal{A}'})$$

Entanglement Entropy

When the system splits into two subsystems, $\mathcal{A} = \mathcal{A}_A \otimes \mathcal{A}_B$, the von Neumann entropy of the reduced density matrix associated to any of them, $\rho_A \equiv \rho_{\omega,\mathcal{A}_A}$, is called the **entanglement entropy**:

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The entanglement entropy is a measure of the degree of entanglement between A and B. If the global state is pure and separable, then ρ_A is pure and $S_{\mathrm{EE}}(A) = 0$.

Summary: entanglement entropy

- Given a density matrix induced by a global state in some algebra, we can compute its von Neumann entropy as $S(\rho) = -\operatorname{tr}(\rho \log \rho)$. The entropy is positive whenever ρ is mixed and it vanishes whenever it is pure.

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- Given a density matrix induced by a global state in some algebra, we can compute its von Neumann entropy as $S(\rho) = -\operatorname{tr}(\rho \log \rho)$. The entropy is positive whenever ρ is mixed and it vanishes whenever it is pure.
- When the system splits into two subsystems A, B, the algebra of B coincides with the commutant of the algebra of A.
- The entanglement entropy of A with respect to B, $S_{\mathrm{EE}}(A)$, is defined as the von Neumann entropy associated to the reduced density matrix on A, and it equals $S_{\mathrm{EE}}(B)$. When the global state is entangled, $S_{\mathrm{EE}} > 0$ and there is entanglement between the two subsystems.

A closer look?

- ⋆ Read about the Schmidt decomposition and how it makes manifest that $S_{\rm EE}(A) = S_{\rm EE}(B)$.
- ⋆ Consider the following global two-qubit states $|\psi_i\rangle \in \mathcal{H}_A \otimes \mathcal{H}_B$ and show that the EE reads in each case:

$$|\psi_1\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |00\rangle\right) \quad \Rightarrow \quad S_{\rm EE}(A) = 0$$

$$|\psi_2\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \quad \Rightarrow \quad S_{\rm EE}(A) = \log 2 \simeq 0.6931,$$

$$|\psi_3\rangle = \frac{1}{\sqrt{3}}\left(|00\rangle + |01\rangle + |11\rangle\right)$$

$$\Rightarrow S_{\rm EE}(A) = \log\left[\frac{6}{3+\sqrt{5}}\right] \simeq 0.1362,$$

Modular Hamiltonian and modular flow

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so it coincides with the usual thermodynamic entropy.

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Consider now a general density matrix. Since it is a positive operator, we can always write it as

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The evolution of operators under this is called **modular flow**, $O(\tau) \equiv U(\tau) O U(-\tau)$, which leaves expectation values invariant, $\mathrm{tr}(\rho O(\tau)) = \mathrm{tr}(\rho O)$.

Summary: modular Hamiltonian

- Any density matrix comes with an associated modular Hamiltonian, $\rho \equiv \mathrm{e}^{-K} / \operatorname{tr} \mathrm{e}^{-K}$. The von Neumann entropy of ρ can be thought of as the canonical entropy for an equilibrium state at temperature 1 for such "Hamiltonian".

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- The time evolution associated to the modular Hamiltonian defines its associated modular flow, $U(\tau) \sim \mathrm{e}^{-i\tau K}$. Expectation values are invariant under it.

A closer look?

- It is always possible to purify a given mixed state ρ_A by enlarging the original Hilbert space $\mathcal{H}_A$ with a copy $\mathcal{H}_B$. In the tensor product, we can define a pure state $|\Omega\rangle$ so that $\rho_A = \text{tr}_B |\Omega\rangle \langle\Omega|$.

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- Provided all the eigenvalues of ρ_A are non-vanishing, the purification defines: the modular operator $\Delta = \rho_A \otimes \rho_B^{-1}$, and the modular conjugation J, which maps the algebra $\mathcal{A}_A$ into its commutant $\mathcal{A}_B$.

A closer look?

- It is always possible to purify a given mixed state ρ_A by enlarging the original Hilbert space $\mathcal{H}_A$ with a copy $\mathcal{H}_B$. In the tensor product, we can define a pure state $|\Omega\rangle$ so that $\rho_A = \text{tr}_B |\Omega\rangle \langle\Omega|$.
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- The modular flow intrinsic to the original algebra $\rho_A^{i\tau}$ can be extended to the purification by $U(\tau) = \Delta^{i\tau}$, which satisfies $U(\tau)\mathcal{A}_A U(-\tau) = \mathcal{A}_A$, $U(\tau)\mathcal{A}_B U(-\tau) = \mathcal{A}_B$.

Relative Entropy

Given two states ω_1, ω_2 and a single algebra $\mathcal{A}$, their **relative entropy** reads (where $\rho_i \equiv \rho_{\omega_i,\mathcal{A}}$):

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The relative entropy is a measure of distinguishability between some reference state and another. The greater the algebra in which we are comparing the states, the more we can distinguish them.

Mutual Information

Given a state ω and two subsystems A,B with algebra $\mathcal{A} = \mathcal{A}_A \otimes \mathcal{A}_B$ we can define the **mutual information** (MI) between both algebras as

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- **Strong Subadditivity**: $I(A,B) \leq I(A, B\cup C)$

The correlations between A and B are always smaller (or equal) than the correlations between A and any enlarged version of B.

Summary: relative entropy and mutual information

- Given two states represented by their density matrices ρ_1, ρ_2 in a single algebra, the relative entropy $S_{\rm rel}(\rho_1|\rho_2) \equiv \text{tr}(\rho_1 \log \rho_1 - \rho_1 \log \rho_2)$ quantifies how distinguishable the states are in that algebra. The greater the algebra, the more we can distinguish them. The relative entropy is always positive except if the states are equal, in which case it vanishes.

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- Given a global state and two algebras $\mathcal{A} = \mathcal{A}_A \otimes \mathcal{A}_B$, the mutual information $I(A,B) \equiv S_{\mathrm{EE}}(A) + S_{\mathrm{EE}}(B) - S_{\mathrm{EE}}(A \cup B)$ quantifies the degree of entanglement shared by the algebras in such state. It is monotonically increasing under inclusions, $I(A,B) \leq I(A, B \cup C)$.

A closer look?

- ⋆ There exists many additional interesting and entanglement-related quantities (Rényi entropy, reflected entropy, negativity, entanglement of purification, etc.). Read about them and find out what makes them special/relevant.

Part II: Entanglement in QFT

Warm up

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- The Hilbert space of states contains a special element, the vacuum $|\Omega\rangle$, which has minimal energy and from which the whole Hilbert space can be accessed by acting on it with linear combinations of products of operators.
- Wightman's reconstruction theorem states that the full information about a QFT is encoded in its vacuum fluctuations:

$$\{\Phi(x), \mathcal{H}\} \quad \Leftrightarrow \quad \langle\Omega| \, \Phi(x_1) \cdots \Phi(x_n) \, |\Omega\rangle$$

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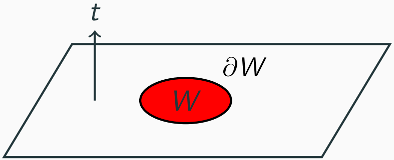

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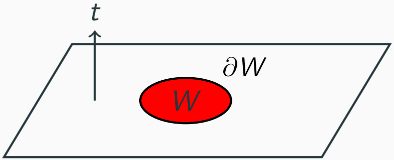

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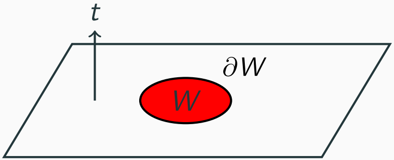

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The “natural” subsystems in QFT are therefore spacetime regions.

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  Operators in V' must commute with operators in the algebra of V:

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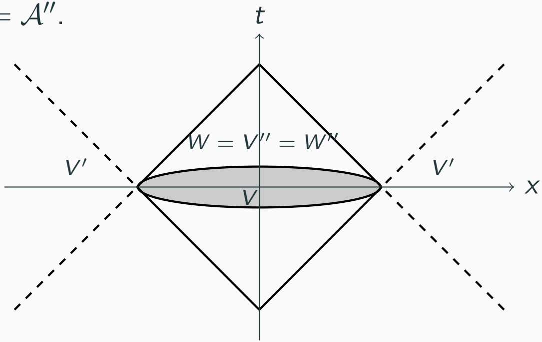

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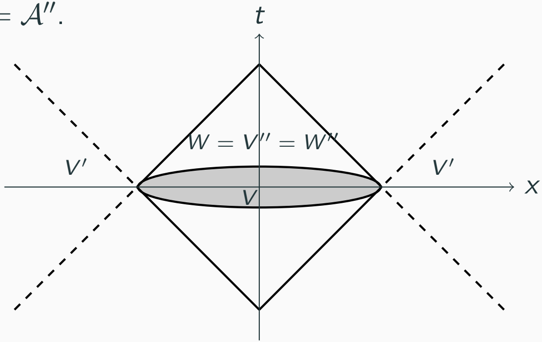

Causal diamonds are the minimal "physical laboratories" in which local quantum experiments could be performed: $\mathcal{A}(V) = \mathcal{A}(W)$

Haag duality

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On the other hand, when it holds for arbitrary regions, the property is called simply **duality**. This takes place only for sufficiently "complete" theories...

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- Open question: is there an algebraic version of the Wightman reconstruction theorem? Can we reconstruct the full information about a QFT from the mutual information of subregions?

$$\{\mathcal{A}(W), \mathcal{H}\} \quad \Leftrightarrow \quad I(V, W)$$

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- An algebraic reconstruction theorem?

$$\{\mathcal{A}(W), \mathcal{H}\} \quad \Leftrightarrow \quad I(V, W)$$

A closer look?

- ⋆ Read about the interplay between violations of the duality relation for regions with non-trivial topology and superselection sectors
  [More in Casini's lectures]

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The theorem follows from the analyticity properties of vacuum correlation functions, which in turn result from the positivity of energy, Lorentz invariance, and locality.

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According to R.S. theorem, $\exists$ some operator a with support in this room such that

$$\langle a\Omega|P|a\Omega\rangle \approx 1$$

namely, such that in that state there is a peanut in Andromeda.

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However, it makes manifest the existence of strong non-local quantum correlations in the vacuum state.

$\langle\Omega|Pa^\dagger a|\Omega\rangle \neq \langle\Omega|P|\Omega\rangle \langle\Omega|a^\dagger a|\Omega\rangle \Leftrightarrow$ Manifest non-separability

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- Any state of the Hilbert space can be approximated by acting with operators on the vacuum $|\Omega\rangle$.
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- The theorem makes manifest the existence of strong non-local quantum correlations in the vacuum state.

A closer look?

- ⋆ Read about the notions of “cyclic” and “separating” states and their connection with the R.S. theorem.
- ⋆ Read about the issues that arise with the notion of “localized states” as a consequence of the R.S. theorem and the role played by “nuclearity conditions”.

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The Hilbert space does not factorize across spacelike-separated regions,

$$\mathcal{H} \neq \mathcal{H}_W \otimes \mathcal{H}_{W'} \quad \text{in QFT}$$

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- Local reduced states are *always* mixed, they cannot be represented by standard density matrices (and cannot be interpreted as statistical mixtures of pure states). Entanglement is irreducible, it cannot be eliminated by local operations.
- The information about the QFT is not in the algebras themselves (they are all isomorphic), but in their relations.

A closer look?

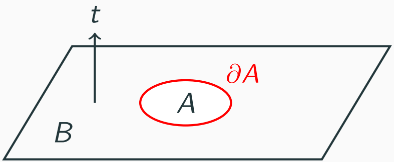
- ⋆ Read about the "split property" and how it helps define approximate notions of Hilbert space factorization in this context.
- ⋆ Read about the " Bisognano-Wichmann theorem" and how it hints at the type-III nature of region algebras in QFT.

A philosophical observation...

...despite its conservative way of dealing with physical principles, algebraic QFT leads to a radical change of paradigm. Instead of the Newtonian view of a space-time filled with a material content, one enters the reality of Leibniz created by relation (in particular inclusions) between "monads" (the hyperfinite type-III local von Neumann factors, which as single algebras are nearly void of physical meaning). (Schroer'98)

(No) entanglement entropy in QFT

Fix some time-slice Σ, divide it in two regions A and B:



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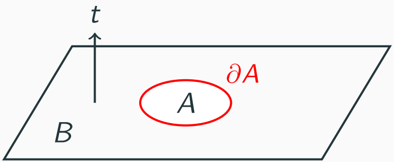

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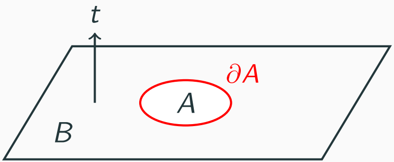

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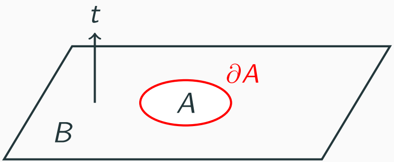

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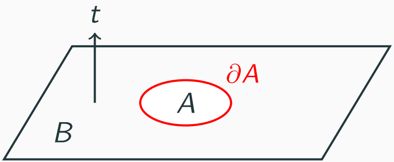

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There is infinite entanglement between any pair of adjacent regions.

A closer look?

⋆ Read about the firewalls proposal (and a putative breakdown of entanglement across the horizon) in the context of the black hole information paradox.

Part III: From the lattice to the continuum

QFTs from the lattice

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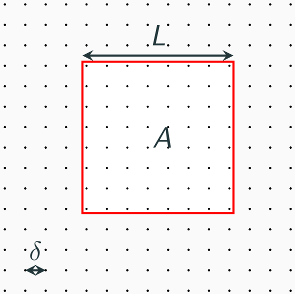
Sometimes it is possible to think of a QFT as the continuum limit of a discrete model. In the limit in which the lattice spacing goes to zero (compared with the relevant physical scales) one would expect to reproduce whatever results may be well-defined in the continuum theory. Such **universal** quantities should be independent of the particular regulator utilized



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This **area law** of EE holds in any state (any state looks like the vacuum at sufficiently short distances).

General structure of EE

Given a QFT$_d$ and a smooth entangling region A, the EE takes the form

$$S_{\mathrm{EE}}^{(d)} = b_{d-2}\frac{L^{d-2}}{\delta^{d-2}} + b_{d-4}\frac{L^{d-4}}{\delta^{d-4}} +$$

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For example, in $d = 2$ theories, for a single interval region of length L,

$$S_{\rm EE}^{(2)} = \frac{c}{3} \log\left(\frac{L}{\delta}\right) + b_0 \,,$$

where c is the Virasoro central charge of the theory.

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For instance, in $d = 4$:

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Similar story in $d = 6, 8, \ldots$

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- Certain terms in such expansions are independent of the way we regulate the theory and capture information about the continuum theory.

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  [More in Moreno and Lasso’s talks]
- ⋆ Read about the “Casini-Huerta-Myers” maps and the tools involved in the proof that $F(\mathbb{B}^{d-1}) = -\log Z_{\mathbb{S}^d}$.

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The evaluation of (regulated) EE and related quantities is in general a very challenging problem in QFT. Roughly, we can divide the existing techniques in “Euclidean” and “Real-time” methods.

Euclidean methods usually make use of an Euclidean path-integral representation of ρ_A. On the other hand, in the real-time approach, one tries to obtain ρ_A in terms of correlators of the fundamental fields.

Here I will give you a flavor of the second type of methods in the case of free fields in the lattice.

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Once we have the eigenvalues of ρ_A, we can rewrite the EE in terms of the correlators using the usual formula ($S_{\mathrm{EE}} = -\,\mathrm{tr}\,\rho_A \log \rho_A$).

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We can diagonalize it to obtain the eigenvalues of ρ_A in terms of M and N. Next, we impose the consistency relations, $X_{ij} = \mathrm{tr}(\rho_A \phi_i \phi_j)$, etc.

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or in terms of the eigenvalues of C,

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The thing is that computing correlators like X_{ij} and P_{ij} (and consequently, C_{ij}) is usually something rather doable, so we can evaluate $S_{\rm EE}(A)$ using the above formula once we know them.

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Using the consistency relations, the eigenvalues of ρ_A can be written in terms of the eigenvalues of C_{ij}, so we can write the EE in terms of the correlators matrix. The result is:

$$S_{\rm EE}(A) = -\,\mathrm{tr}[(1-C)\log(1-C) + C\log C]$$

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$$C_{jl} \equiv \langle \psi_j \psi_l^\dagger \rangle = \begin{cases} \frac{(-1)^{(j-l)}-1}{2\pi i (j-l)} & j \neq l \\ \frac{1}{2} & j = l \end{cases} .$$

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For a general CFT_2, the result for the EE of an interval reads

$$S^{(2)}_{\text{EE}} = \frac{c}{3} \log(L_A/\delta) + \mathcal{O}(\delta^0)$$

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For technical reasons ("fermion doubling"), when performing lattice calculations for fermions in $d = 2$ one gets an extra factor of 2. A small program in Mathematica yields perfect agreement:

```
In[258]:= c[x_] := If[x == 0, 1/2, N[((-1)^(x) - 1) / (2 Pi I x)]]
                   |si        |valor numérico      |··· |número i

In[259]:= entro[reg_] :=
           Module[{corr, v},
           |módulo
            corr = Table[c[reg[[i]] - reg[[j]]], {i, 1, Length[reg]}, {j, 1, Length[reg]}];
                   |tabla                                 |longitud                |longitud
            v = Re[Eigenvalues[corr]];
                |pa· |autovalores
            Re[-v.Log[v + 10^(-11)] - (1 - v).Log[1 - v - 10^(-11)]]]
            |parte ··· |logaritmo            |logaritmo

In[270]:= entropia = Table[entro[Table[j, {j, 1, i * 10}]], {i, 1, 25}]
                     |tabla        |tabla

Out[270]= {1.49342, 1.7246, 1.85978, 1.95568, 2.03007, 2.09084, 2.14223, 2.18674,
           2.226, 2.26112, 2.29289, 2.3219, 2.34858, 2.37328, 2.39628, 2.41779, 2.438,
           2.45705, 2.47507, 2.49217, 2.50844, 2.52394, 2.53876, 2.55295, 2.56655}

In[271]:= Fit[entropia, {Log[x], 1}, x]
          |ajusta        |logaritmo

Out[271]= 1.49351 + 0.333363 Log[x]
```

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- The exist many different approaches which allow us to evaluate the EE of region algebras in certain cases.
- I have briefly explained one: the real-time method for free fields, which allows one to evaluate it in terms of expectation values of the fundamental fields, which are usually much easier to compute in explicit models (and particularly suitable for the lattice).

A closer look?

- ⋆ Write your own code (or use mine) for computing the EE of $d = 2$ free fermions in the lattice and try to reproduce the $c = 1/2$ result. Explore the dependence of the mutual information of pairs of intervals as a function of their distance using the same code.

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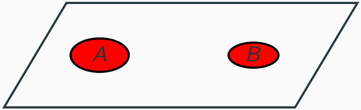
Mutual information	✓	Well defined (using relative entropy). Using regulated EE definition, all divergences cancel each other in the continuum limit

Part IV: QFT from entanglement

Mutual information in QFT

As opposed to the entanglement entropy, the mutual information of pairs of region algebras is well defined in a QFT.



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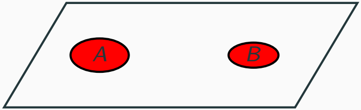

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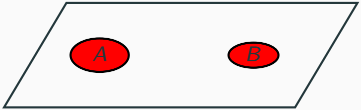

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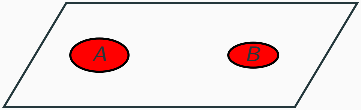
As opposed to the entanglement entropy, the mutual information of pairs of region algebras is well defined in a QFT.



From the regulated theory point of view, all divergences appearing in

$$I(A, B) = S_{\mathrm{EE}}(A) + S_{\mathrm{EE}}(B) - S_{\mathrm{EE}}(A \cup B)$$

exactly cancel each other in the continuum limit.

On the other hand, it can be rigorously defined directly in the continuum using its definition in terms of the relative entropy

Tripartite information in QFT

Generalization to three regions: **tripartite information**

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$$I_N(A_1, A_2, \dots, A_N) \equiv -\sum_{\sigma}(-1)^{\#_\sigma} S_{\text{EE}}(\sigma)\,, \quad \sigma \subset \{A_1, A_2, \dots\}$$
$$= I_{N-1}(\star, A_{N-1}) + I_{N-1}(\star, A_N) - I_{N-1}(\star, A_{N-1} \cup A_N)\,,$$

where $\star \equiv A_1, \dots, A_{N-2}$. It measures the non-extensivity of I_{N-1}.

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- The MI for pairs of non-intersecting region algebras is well-defined in QFT. If we use a regulator, all the EE divergences cancel each other in the continuum limit.
- Using the MI as a building block, we can define associated multi-partite notions for an arbitrary number of regions as

$$I_N = I_{N-1}(\star, A_{N-1}) + I_{N-1}(\star, A_N) - I_{N-1}(\star, A_{N-1} \cup A_N)\,,$$

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QFT formulations and reconstruction theorems

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Given $I(A, B)$ for all regions, how do we reconstruct the theory?

Mutual information as a geometric regulator

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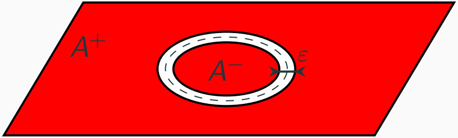

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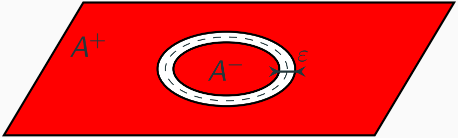

Then, the EE of A can be approximated as

$$S_{\mathrm{EE}}^{(\varepsilon)}(A) \approx \frac{1}{2} I_\varepsilon(A^+, A^-)\,, \qquad (\varepsilon \ll L_A)$$

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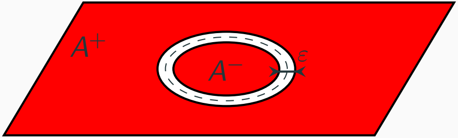

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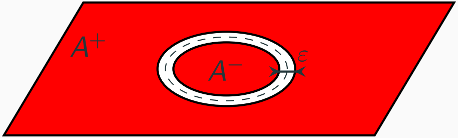

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All EE universal terms are robustly captured by the MI. This is not just a technical curiosity: it becomes crucial in certain situations (e.g., for general odd-dimensional QFTs or for orbifold theories).

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On general grounds, the MI satisfies the bound

$$I(A,B) \geq \frac{1}{2}\frac{\langle \mathcal{O}_A \mathcal{O}_B \rangle_{\rm c}^2}{||\mathcal{O}_A||^2||\mathcal{O}_B||^2}\,,$$

for any operators $\mathcal{O}_A$, $\mathcal{O}_B$ supported in A and B, respectively.

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where $f(d,\Delta)$ is a theory dependent quantity which also depends on the shape of A and B.

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Subleading terms in the long-distance expansions will produce the rest of dimensions and OPE coefficients. Doing this explicitly is easier said than done...

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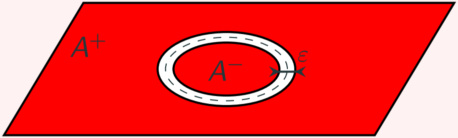

A closer look?

- ⋆ Read about the "Réplica trick" and "twist operators", and the role they play in the long-distance expansion of mutual information.

  [More in Takayanagi's lectures]

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- For A, B sharing planar boundary proportional to η
  $I(A, B + \epsilon\eta) \sim \epsilon^{-(d-2)}$ as $\epsilon \to 0$ (Area law)

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Some solutions correspond to scaled limits of CFTs, e.g., the Ryu-Takayanagi formula for large-N CFTs with Einstein gravity duals

$$S^{\text{holo}}(A) = \underset{\Gamma_A \sim A}{\text{ext}} \left[\text{Area}\left(\frac{\Gamma_A}{4G} \right) \right] + \ldots$$

[More in Takayanagi's lectures]

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Interestingly, added to the axioms, this leads to a closed explicit geometric formula

$$I_2^{\mathrm{EMI}}(A, B) = 2\kappa_{(d)} \int_{\partial A} \mathrm{d}\sigma_A \int_{\partial B} \mathrm{d}\sigma_B \frac{n_A \cdot n_B}{|x_A - x_B|^{2(d-2)}}$$

where $\kappa_{(d)}$ is a constant characterizing the model.

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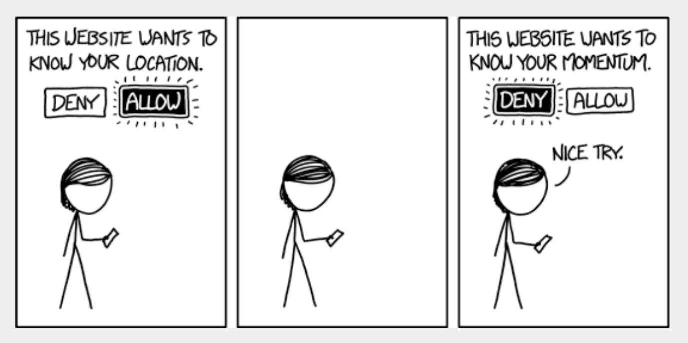
A closer look?

- ⋆ Do not forget to go to the Giambiagi school webpage, download these slides and have a closer look at them!

The End


THIS WEBSITE WANTS TO KNOW YOUR LOCATION.
DENY
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BONUS

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- Then, the N-partite information can be written as

$$I_N(A_1, \ldots, A_N) = \lim_{n \to 1} \frac{(-)^{N+1}}{1-n} \langle \tilde{\Sigma}_{A_1}^{(n)} \tilde{\Sigma}_{A_2}^{(n)} \cdots \tilde{\Sigma}_{A_N}^{(n)} \rangle$$

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- For spherical entangling surfaces,

$$C_{ij}^A = \frac{R^{2\Delta}}{\sin^{2\Delta}\left[\frac{\pi(i-j)}{n}\right]}$$

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$$I_3(A_1, A_2, A_3) = -\lim_{n\to 1} \frac{1}{n-1} \langle \tilde{\Sigma}^{(n)}_{A_1} \tilde{\Sigma}^{(n)}_{A_2} \tilde{\Sigma}^{(n)}_{A_3} \rangle$$

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which appear written in terms of correlators of $\mathcal{O}$:
$\langle \mathcal{O}_{A_1} \mathcal{O}_{A_2} \rangle = \frac{1}{L_{12}^{2\Delta}}$, $\quad \langle \mathcal{O}_{A_1} \mathcal{O}_{A_2} \mathcal{O}_{A_3} \rangle = \frac{C_{\mathcal{O}\mathcal{O}\mathcal{O}}}{L_{12}^{\Delta} L_{13}^{\Delta} L_{23}^{\Delta}}$

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Recall long-distance expression for a CFT whose lowest-dim operator is a fermion:

$$I_2^f = 2^{\lfloor\frac{d}{2}\rfloor+1}\frac{\sqrt{\pi}\Gamma[2\Delta+1]}{4\Gamma[2\Delta+\frac{3}{2}]}[2(n_A\cdot\ell)(n_B\cdot\ell)-(n_A\cdot n_B)]\cdot\frac{R_A^{2\Delta}R_B^{2\Delta}}{L^{4\Delta}}+\dots$$

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We learn that the EMI model necessarily contains a free fermion ($\Delta = (d-1)/2$) as its lowest-dim operator